UNIT 2 SIMPLE APPLICATION OF DIFFERENTIAL CALCULUS

Structure

2.0 INTRODUCTION

In the previous unit, we defined the derivative of a function. In this Unit, we shall discuss several applications of the derivative of a function. Derivatives have been used in various spheres of Physics, Chemistry, Medical Sciences and Economics.

Derivative of y with respect to x determines the rate of change of y with respect to x. This can be used to find rate of change of a quantity with respect to another quantity. Chain rule can be used to find rates of change of two quantities that are changing with respect to the same quantity.

The concept of derivative can also be used to determine whether a function is increasing or decreasing in an interval. Indeed, if the derivative of a function is positive in an interval, then the function increases in that interval. Similarly, if the derivative is negative then the function decreases. If the derivative is zero in an interval, then the function remains constant in that interval.

Derivatives can also be used to find maximum and minimum values of a function in an interval. The maximum and minimum values are called extreme values of a function. The extreme values can be absolute or can be local. The first derivative test and the second derivative tests are used to determine the points of local extrema.

2.1 OBJECTIVES

After studying this unit, you should able to :

- compute the rate of change of one quantity, knowing the rate of change of some other related quantity;

- determine when a function increases or decreases;
- find the absolute maximum and absolute minimum of a continuous function on the closed interval $[a,b]$; and
- find local maximum and local minimum of a function.

2.2 RATE OF CHANGE OF QUANTITIES

Let us consider a differentiate function $y = f(x)$. We can give a physical interpretation to $\frac{dy}{dx}$ the derivative $\frac{dy}{dx} = f'(x)$ represents the rate of change of y with respect to x, when y varies with x according to the rule $y = f(x)$.

Thus, $f'(x) \left(or \left.\frac{dy}{dx}\right]_{x=x_0} \right)$ represents the rate of change of y with respect to x at $x = x_0$.

For example, Let s be the distance of a particle from origin at a time t. Then s is a function of t and the derivative $\frac{ds}{dt}$ represents speed which is the rate of change of distance with respect to time t.

We can use chain rule to find the rates of two or more related variables that are changing with respect to the same variable. Suppose x and y are both varying with t (i.e., x, y are function of t). Then, by chain rule, we have

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = f'(x)\frac{dx}{dt}$$

Thus, the rate of change one variable can be calculated if the rate of change of the other variable is known.

Example 1: A spherical balloon is being inflated at the rate of 900 cubic centimeters per second. How fast is the radius of the balloon increasing when the radius is 15 cm ?

Solution : We are given the rate of change of volume and are asked the rate of change of the radius. Let r cm be the radius and V cubic centimeters be the volume of the balloon at instant t.

$$\text{Thus, } V = \frac{4}{3}\pi r^3$$

Differentiating both the sides of this equation with respect to t, we get

$$\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt} = 4\pi r^2\frac{dr}{dt}.$$

Note that have used the chain rule on the right hand side

It is given that $\frac{dV}{dt} = 900$ and $r = 15$. Thus,

$$900 = 4\pi\,(15)^2\,\frac{dr}{dt}$$

$$\Rightarrow \frac{dr}{dt} = \frac{900}{4\pi\,(225)} = \frac{1}{\pi}\ cm/\,sec.$$

$$\text{Thus, the radius is increasing at the rate of } = \frac{1}{\pi}\ \ cm/\,sec.$$

Example 2 : A rock is thrown into a lake producing a circular ripple. The radius of the ripple is increasing at the rate of 3 m/s. How fast is the area inside the ripple increasing when the radius is 10 m ?

Solution : We are given the rate of change of the radius and we are asked to find the rate of change of area.

Let r m be the radius and A square meters be the area at time t. Thus, $A = \pi\, r^2$.

Differentiating both the sides with respect to t, we get

$$\frac{dA}{dt} = \frac{dA}{dr}\,.\frac{dr}{dt} = 2\,\pi\, r\ \frac{dr}{dt}$$

We are given when $r\ =\ 10, \frac{dr}{dt} = 3$, therefore $\frac{dA}{dt} = 2\,\pi(10)(3) = 60\,\pi$

Thus, the area is increase at the rate of 60 $\pi\ sq.m/s$.

Example 3 : If a mothball evaporates at a rate proportional to its surface area 4 πr^2, show that its radius decreases at a constant rate.

Solution : Let r be the radius V be the volume of the mouthball at time t. Then

$$V = \frac{4}{3}\pi r^3$$

We are given that the mothball evaporates at a rate proportional to its surface area. It means that the rate of decrease of volume V of the mothball is proportional to 4 πr^2.

$$\text{Thus,} \frac{dV}{dt}\ = -\,k\ (4\pi r^2)$$

where $k > 0$ is a constant. (Negative sign has been introduced to show that the volume is decreasing).

$$\text{But}\ \ \frac{dV}{dt} = 4\pi r^2\ \ \frac{dr}{dt}.\ \ \text{Threfore}\ 4\pi r^2\,\frac{dr}{dt} =\ -k\ (4\pi r^2)\, or \frac{dr}{dt} =\ -k$$

This shows that the radius decreases at a constant rate. [Decrease is due to the negative sign].

Example 4 : A young child is flying kite which is at a height of 50 m. The wind is carrying the kite horizontally away from the child at a speed of 6.5 m/s. How fast must the kite string be let out when the string is 130 m ?

Solution : Let h be the horizontal distance of the kite from the point directly over the child's head 5. Let l be the length of kite string from the child to the kite at time t. [See Fig. 1] Then

$$l^2 = h^2 + 50^2$$

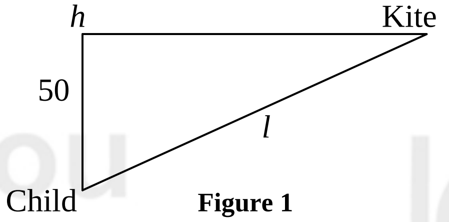

Figure 1

Differentiating both the sides with respect to t, we get

$$2l\frac{dl}{dt} = 2h\frac{dh}{dt} \text{ or } l\frac{dl}{dt} = h\frac{dh}{dt}.$$

We are given $\frac{dh}{dt} = 6.5\ m/s.$ We are interested to find dl/dt when $l = 130$.
But when $l = 130$, $h^2 = l^2 - 50^2 = 130^2 - 50^2 = 14400$ or $h = 120$.

$$\text{Thus, } \frac{dl}{dt} = \frac{120}{130} \times 6.5 = 5\ m/s.$$

This shows that the string should be let out at a rate of 6 m/s.

Example 5 : A ladder 13 m long leans against a house. The foot of the ladder is pulled along the ground away from the wall, at the rate of 1.5 *m/s*. How fast is the angle ∝ between the ladder and the ground is changing when the foot of ladder is 12 m away from the house.

Solution : Let y be the distance of the top of the ladder from the ground, and let x be the distance of the bottom of the ladder from the base of the wall at time (Figure 2) t. We have $\tan \alpha = \frac{y}{x}$

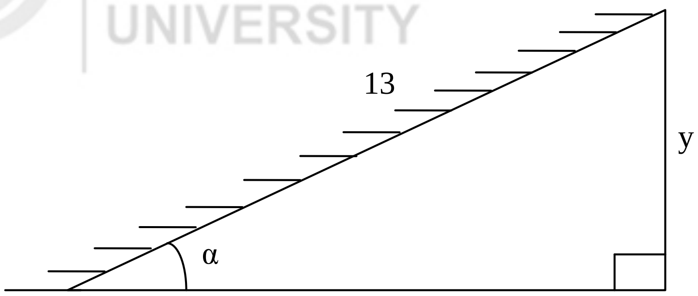

Figure 2

Differentiating both the sides with respect to *t*, we get

$$Sec^2\,\alpha\,\frac{d\alpha}{dt} = \frac{x\frac{dy}{dt} - y\,\frac{dx}{dt}}{x^2}$$

Now by the Pythagorean theorem $x^2 + y^2 = 13^2$

Differentiating both the sides with respect to $t,$ we get $\quad 2x\frac{dx}{dt} + 2y\,\frac{dy}{dt} = 0$

$$\Rightarrow \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

$$\text{Thus,} \frac{d\alpha}{dt} = \frac{\frac{-x^2}{y}\frac{dx}{dt} - y\,\frac{dx}{dt}}{x^2 \text{Sec}^2\,\alpha} = - \frac{x^2+y^2}{x^2\,y[1+tan^2\alpha]}\frac{dx}{dt}$$

$$= -\frac{x^2+y^2}{x^2\,y\left[1+\frac{y^2}{x^2}\right]}\;\frac{dx}{dt} = -\frac{1}{y}\;\frac{dy}{dt}$$

we are given that $\frac{dx}{dt} = 1.5$ and $x = 12.$ For $x = 12,$ we have

$$y = \sqrt{13^2 - x^2} = \sqrt{13^2 - 12^2} = 5$$

$$\text{Therefore,} \frac{d\alpha}{dt} = -\frac{1}{5}\;(1.5) = -\;0.3$$

Thus, α is decreasing at the rate of 0.3 radians per second.

Example 6 : Water, at the rate of 15 cubic centimetres per minute is pouring into a leaking cistern whose shape is a cone 16 centimetres deep and 8 centimetres in diameter at the top. At the time, the water is 12 centimetres deep, the water level is observed to be rising 0.5 centimetres per minutes. How fast is the water leaking out ?

Solution : Let *h* be the depth of the water, and *r* be the radius of the water surface at time *t*. [see fig 3]. Let *V* be the volume of the water at time *t*,

Then $\qquad V = \frac{1}{3}\pi r^2 h$

Now, since ΔOAB is similar to ΔOCD, we have

$$\frac{AB}{OA} = \frac{CD}{OC} \Rightarrow \frac{r}{h} = \frac{4}{16} \text{ or } r = \frac{1}{4}\,h.$$

$$\text{Thus,} V = \frac{1}{3}\,\pi\left(\frac{1}{4}\,h\right)^2 h = \frac{1}{48}\,\pi\,r^3$$

Differentiating both the sides with respect to *t*, we get

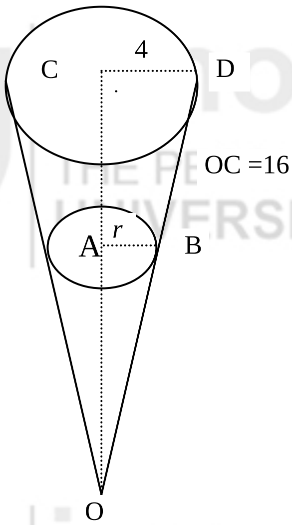

Figure 3

$$\frac{dV}{dt} = \frac{1}{48}\,\pi(3\,h^2)\frac{dh}{dt} = \frac{\pi h^2}{16}\frac{dh}{dt}$$

We are given that when $h = 12, \frac{dh}{dt} = \frac{1}{2}$ Hence, at that moment,

$$\frac{dV}{dt} = \frac{\pi(12)^2}{16} \times \frac{1}{2} = 4.5\,\pi.$$

Since, the rate at which the water is pouring in is 15, the rate of leakage is $(15 - 4.5\pi)$ cm^3 /min.

Example 7 : Sand is being poured into a conical pile at the constant rate of 50 cubic centimetres per minute. Frictional forces in the sand are such that the height of the cone is always one half of the radius of its base. How fast is the height of the pile increasing when the sand is 5 cm deep ?

Solution : Let r the radius, h be height and V be the volume of the cone (Fig. 4) of sand at time t.

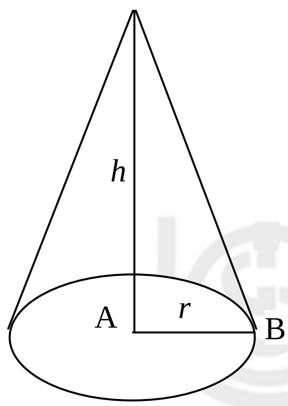

Figure 4

Then $V = \frac{1}{3}\pi r^2 h$

We are given that $h = \frac{1}{2} r$ or $r = -2h.$ Thus,

$V = \frac{1}{3}\pi(2h)^2 h = \frac{4}{3}\pi h^3.$

Differentiating both the sides with respect to t, we get

$$\frac{dV}{dt} = \frac{4}{3}\,\pi(3\,h^2)\frac{dh}{dt} = 4\pi h^2\frac{dh}{dt}.$$

we are given that $\frac{dV}{dt} = 50$, thus $50 = 4\pi h^2\frac{dh}{dt}.$

$$\Rightarrow \frac{dh}{dt} = \frac{50}{4\pi h^2}$$

When $h = 5. \frac{dh}{dt} = \frac{50}{4\pi(5^2)} = \frac{50}{100\pi} = \frac{1}{2\pi}.$

Hence, the height of the cone is rising at the rate of $(1/2\pi)$ cm/min.

Check Your Progress – 1

1. The side of a square is increasing at the rate of 0.2 cm/s. Find the rate of increase of the perimeter of the square.
2. The radius of a circle is increasing at the rate of 0.7 cm/s. What is the rate of increase of its circumference ?

3. A man 180 cm tall walks at a rate of 2 m/s away from a source of light that is 9 m above the ground. How fast is the length of his shadow increasing when he is 3 m away from the base of light ?
4. A small funnel in the shape of a cone is being emptied of fluid at the rate of 12 cubic centimeters per second. The height of the funnel is 20 centimetres and the radius of the top is 4 centimeters. How fast is the fluid level dropping when the level stands 5 centimetres above the vertex of the cone ?

2.3 INCREASING AND DECREASING FUNCTIONS

In this section, we shall study how the derivative can be used to obtain the interval in which the function is increasing or decreasing. We begin with the following definitions.

Definition : A function f is said to be **increasing*** on an interval I if for any two number x_1 and x_2 in I.

$$x_1 < x_2 \text{ implies } f(x_1) < f(x_2).$$

A function f is said to be **decreasing**** on an interval I if for any two numbers x_1 and x_1 in I,

$$x_1 < x_2 \text{ implies } f(x_1) > f(x_2).$$

From this definition, we can see that a function is increasing if its graph moves up as x moves to the right and a function is decreasing if its graph moves down as x moves to the right. For example, the function in Fig 5 is decreasing on the interval $(-\infty, a]$ is constant on the interval $[a,b]$ and is increasing on the interval $[b, \infty)$.

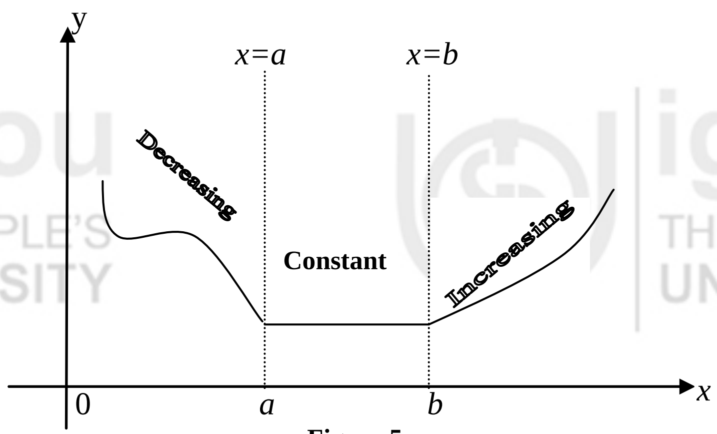

Figure 5

* some authors call such a function strictly increasing.
** some authors call such a function strictly decreasing.

Example 8 : Show that $f(x) = 3x + 2$ is an increasing function on **R**.

Solution :

(i) Let $x_1, x_2 \in \mathbf{R}$ and suppose $x_1 < x_2$
Since $x_1 < x_2$, we have $3x_1 < 3x_2$ ($\because 3>0$)

$\Rightarrow 3x_1 + 2 < 3x_2 + 2 \Rightarrow f(x_1) < f(x_2)$.

Thus, f is increasing on **R.** See also Figure 6.

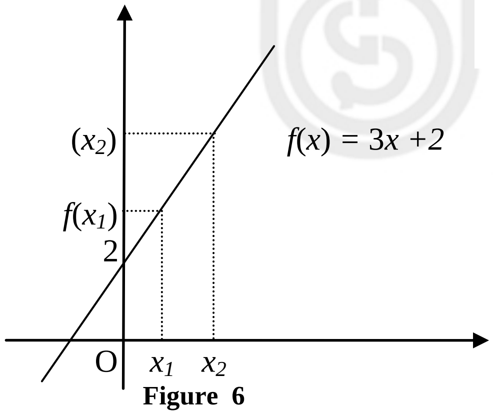

Figure 6

Example 9 : Show that $f(x) = x^2$ is a decreasing function on the interval $(-\infty, 0]$.

Solution : Note that the domain of f consist of all non–positive real numbers.

Suppose $x_1 < x_2 \leq 0$.

Since $x_1 < x_2$ *and* $x_1 < 0$, it follows that $x^2_1 > x_1 x_2$ ……… (1)

Again $x_1 < x_2$ *and* $x_1 \leq 0$, it follows that $x_1 x_2 \geq x_2^2$………..(2)

Combining (1) and (2) we get $x^2_1 > x_1 x_2 \geq x_2^2$

i.e., $x^2_1 > x^2_2$.

which means $f(x_1) > f(x_2)$

Hence, f is a decreasing function on $(-\infty, 0)$. See also Fig 7

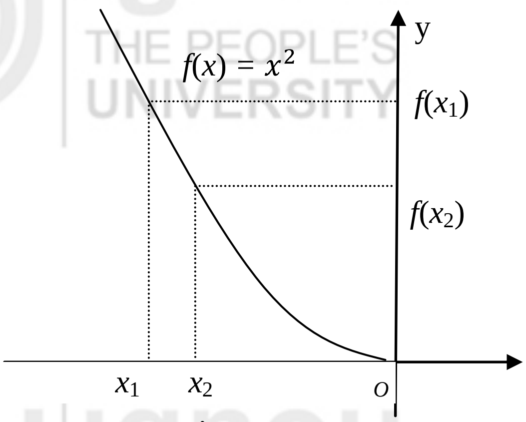

Figure 7

Example 10 : Show that $f(x) = x^2$, $x \in \mathbf{R}$ is neither increasing nor decreasing.

Solution : Let's take two points x_1 and x_2 such that $x_1 < 0 < x_2$. Then $f(x_1) = x^2_1 > 0 = f(0)$ *and* $f(0) = 0 < x^2_2 = f(x_2)$. Since $x_1 < 0$ and $0 < x_2$ implies $f(x_1) > f(0)$ *and* $f(0) < f(x_2)$, it follows that f is neither decreasing nor increasing. See also Fig 8.

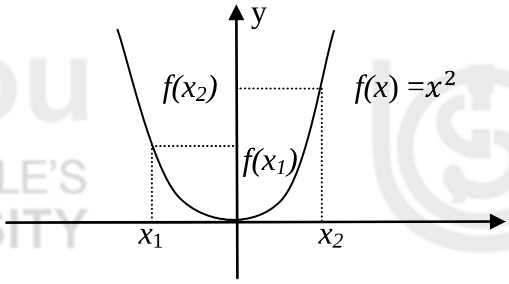

Figure 8

Use of Derivative to check Increasing, Decreasing

We now see how can use derivative f' to determine where a function f is increasing and where it is decreasing.

The graph of $y = f(x)$ in Figure 9 indicates that if the slope of a tangent line is positive in an open interval I (that is, if $f'(x) > 0$ for every $x \in I$), then f is increasing on I. Similarly, it appears that if the slope is negative (that is, $f'(x) < 0$), then f is decreasing

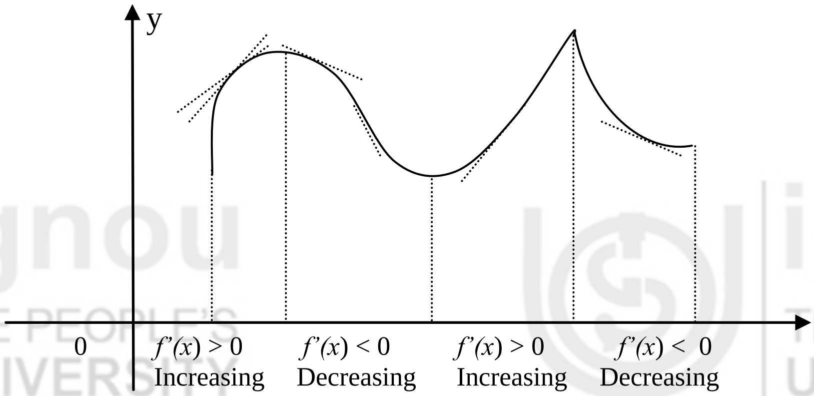

Figure 9

What we have observed intuitively are actually true as a consequence of the following theorem.

Theorem : Let f be continuous on $[a,b]$ and differentiable on (a, b)

(1) If $f'(x) > 0$ for all $x \in (a,b)$, then f is increasing on $[a,b]$.

(2) If $f'(x) > 0$ for all $x \in (a,b)$, then f is decreasing on $[a,b]$.

We may also note that if $f'(x) > 0$ throughout an infinite interval $(-\infty, a]$ or $[b, \infty)$, then f is increasing on $(-\infty, a]$ or $[b, \infty)$, respectively, provided f is continuous on these intervals. An analogus result holds for decreasing function if $f'(x) < 0$.

Example 11 : Determine for which values of x the following functions are increasing and for which values they are decreasing.

(i) $f(x) = 16x^2 + 3x + 2$ (ii) $f(x) = -5x^2 + 7x + 8$

(iii) $f(x) = x^3 - 3x$ (iv) $f(x) = -2x^3 + 24x + 7$

Solution :

(i) We have $f'(x) = 32x + 3$.

Now, $f'(x) > 0$ if $32x + 3 > 0$, that is, if $x > -3/32$ and $f'(x) < 0$ if $32x + 3 < 0$ that is, if $x < -3/32$. Thus, $f(x)$ decreases on $(-\infty, -3/32]$ and increases on $[-3/32, \infty]$.

(ii) We have $f'(x) = -10x + 7$.

Now, $f'(x) > 0$ if $-10x + 7 > 0$, that is, if $x < 7/10$ and $f'(x) < 0$ if $-10x + 7 < 0$, that is, if $x > 7/10$. Thus, $f(x)$ decreases on $[7/10, \infty)$ and increases on $(-\infty, 7/10]$.

(iii) We have $f'(x) = 3x^2 - 3 = 3(x + 1)(x - 1)$.

See figure 10 and note that if $x < -1$, then both $x + 1$ and $x - 1$ are negative and therefore, $f'(x) > 0$.

If $-1 < x < 1$, then $x + 1 > 0$ and $x - 1 < 0$, therefore $f'(x) < 0$. Finally, if $x > 1$, then both $x + 1$ and $x - 1$ are positive and therefore $f'(x) > 0$.

Thus, $f(x)$ increases on $(-\infty, -1] \cup [1, \infty)$ and decrease on $[-1, 1]$.

Sign of $(x+1)$ $- - -$ $+ + +$ $+ + +$

Sign of $(x-1)$ $- - -$ $- - -$ $+ + +$

-1 1 x

$f'(x)$ $+ + +$ 0 $- - -$ 0 $+ + +$

Figure 10

(iv) We have $f'(x) = -6x^2 + 24 = -6(x + 2)(x - 2)$.

Sign of -6 $- - -$ $- - -$ $- - -$

Sign of $(x+1)$ $- - -$ $+ + +$ $+ + +$

Sign of $(x-1)$ $- - -$ $- - -$ $+ + +$

-2 2 x

$f'(x)$ $- - -$ 0 $+ + +$ 0 $- - -$

$f(x)$

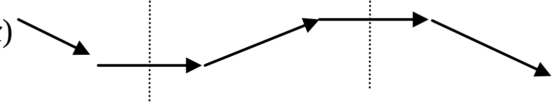

Figure 11

See figure 11 and note that

if $x < -2$ then $x + 2 < 0$, $x - 2$, $0 < 0$ and therefore $f'(x) < 0$.

if $-2 < x < 2$ then $x + 2 > 0$, $x - 2 < 0$, $-6 < 0$ and therefore $f'(x) > 0$.

if $x > 2$, then $x + 2 > 0$, $x - 2 > 0$, $-6 < 0$ and therefore $f'(x) < 0$.

Thus, $f(x)$ decreases on $(-\infty, -2] \cup [2, \infty)$ and increases on $[-2, 2]$.

Example 12 : Determine the values of x for which the following functions are increasing and for which they are decreasing.

(i) $f(x) = x^4 - 8x^3 + 22x^2 - 24x + 21$

(ii) $f(x) = (x-1)(x-2)^2$

(iii) $f(x) = (x-4)^3(x-3)^2$

Solution : (i) We have $f'(x) = 4x^3 - 24x^2 + 44x - 24$

$$= 4(x^2 - 6x^2 + 11x - 6)$$

$$= 4(x-1)(x-2)(x-3)$$

Sign of $(x-1)$	$- - -$	$+ + +$	$+ + +$	$+ + +$
Sign of $(x-2)$	$- - -$	$- - -$	$+ + +$	$+ + +$
Sign of $(x-3)$	$- - -$	$- - -$	$- - -$	$+ + +$

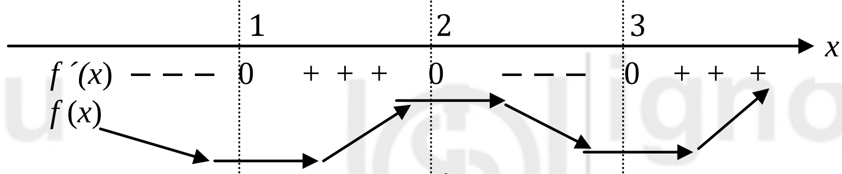

Figure 12

From Figure12 it is clear that

if $x < 1$, then $x - 1 < 0$, $x - 2 < 0$, $x - 3 < 0$ and therefore, $f'(x) < 0$.

if $1 < x < 2$, then $x - 1 > 0$, $x - 2 < 0$, $x - 3 < 0$ and therefore, $f'(x) > 0$,

if $2 < x < 3$, then $x - 1 > 0$, $x - 2 > 0$, $x - 3 < 0$ and therefore, $f'(x) < 0$,

if $x > 3$, then $x - 1 > 0$, $x - 2 > 0$, $x - 3 > 0$ and therefore, $f'(x) > 0$.

Thus, $f(x)$ increases on $[1,2] \cup [3,\infty)$ and decreases on $(-\infty,1] \cup [2,3]$.

(ii) We have $f'(x) = (1)(x-2)^2 + (x-1)(2)(x-2)$

$$= (x-2)(x-2+2x-2) = (x-2)(3x-4) = 3(x - 4/3)(x-2).$$

Sign of $(x - 4/3)$	$- - -$	$+ + +$	$+ + +$
Sign of $(x - 2)$	$- - -$	$- - -$	$+ + +$

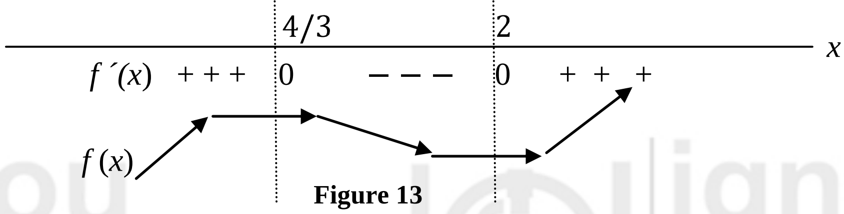

Figure 13

From Figure 13 we note that

if $x < 4/3$, then $x - 4/3 < 0$, $x - 2 < 0$ and therefore, $f'(x) > 0$.

if $4/3 < x < 2$, then $x - 4/3 > 0$, $x - 2 < 0$ and therefore $f'(x) < 0$.

if $x > 2$, then $x - 4/3 > 0$, $x - 2 > 0$ and therefore $f'(x) > 0$.

Thus, $f(x)$ increase on $(-\infty, 4/3] \cup [2, \infty)$ and decreases on $[4/3, 2]$.

(iii) We have $f'(x) = 3(x-4)^2(x-3)^2 + (x-4)^3(2)(x-3)$

$= (x-4)^2(x-3)\,[3(x-3) + 2\,(x-4)\,]$

$= (x-4)^2\,(x-3)\,(5x-17) = 5\,(x-4)^2(x-3)\,(\,x - 17/5)$

Since $(x-4)^2 > 0$ for each $x \neq 0$ the sign of $f'(x)$ depends on the sign of $(x-3)(x-17/5)$.

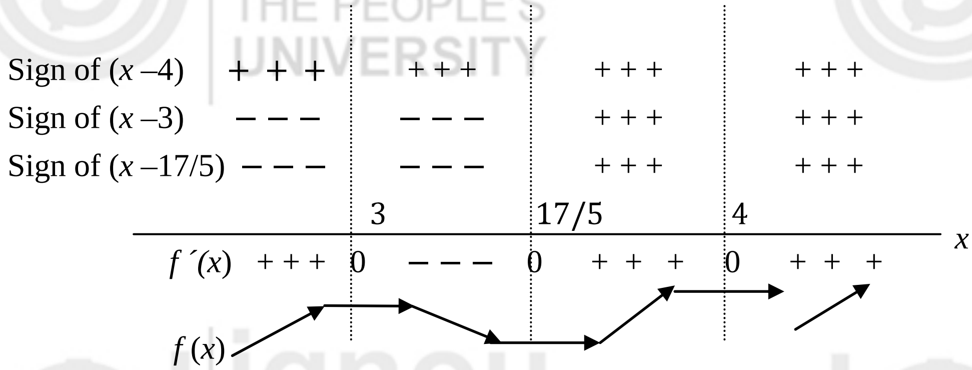

Figure 14

From figure 14 we note that

if $x < 3$, then $x - 3 < 0$, $x - 17/5 < 0$ and therefore, $f'(x) > 0$

if $3 < x < 17/5$, then $x - 3 > 0$, $x - 17/5 < 0$ and therefore $f'(x) < 0$

if $x > 17/5$, $x \neq 4$, then $x - 3 > 0$. $x - 17/5 > 0$ and therefore, $f'(x) > 0$.

Also $f'(4) = 0$.

Thus, $f\ (x)$increase on $(-\infty, 3], [17/5, 4]$and $[4, \infty)$and decreases $[3, 17/5]$.

Hence, $f(x)$increase on $(-\infty, 3] \cup [17/5, \infty)$ and decreases on $[3, 17/5]$.

Example 13 : Determine the intervals in which the following functions are increasing or decreasing.

(i) $f(x) = e^{1/x}, x \neq 0$

(ii) $f(x) = \dfrac{1 + x + x^2}{1 - x + x^2}$, $x \in \mathbf{R}$

Solution :

(i) We have, for $x \neq 0$ $f'(x) = e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\, e^{\frac{1}{x}}$

As $e^{1/x} > 0$ and $x^2 > 0\ \forall\, x \neq 0$, we get $f'(x) < 0\ \forall\, x \neq 0$.

Thus, $f(x)$ decreases on $(-\infty, 0) \cup (0, \infty)$.

(ii) We have $f'(x) = \dfrac{(1+2x)(1-x+x^2)-(-1+2x)(1+x+x^2)}{(1-x+x^2)^2}$

$$= \frac{2(1-x^2)}{(1-x+x^2)^2}$$

Since $(1-x+x^2)^2 > 0$ $f'(x) > 0$ if $1 - x^2 > 0$ or $-1 < x < 1$ and $f'(x) < 0$ if $1 - x^2 < 0$ that is, if $x < -1$ or $x > 1$.

Thus, $f'(x)$ increases on $[-1,1]$ and decreases on $(-\infty, -1] \cup [1,\infty)$.

Check Your Progress – 2

1. Show that $f(x) = x^2$ is a decreasing function on the interval $[0, \infty)$ using the definition of decreasing function.
2. Let $f(x) = ax + b$, where a and b are real constants. Prove that
   (i) If $a > 0$, then f is an increasing function on **R**.
   (ii) If $a < 0$, then f is an decreasing function on **R**.
3. Determine for which values of x the function $f(x) = 2x^3 - 15x^2 + 36x + 1$ is increasing and for which it is decreasing.

4. Determine for which values of x the following functions are increasing and for which they are decreasing.
   (i) $f(x) = x^8 + 6x^2$ (ii) $f(x) = \frac{x}{1+x^2}$
   (iii) $f(x) = x^4 - 4x$ (iv) $f(x) = (x-1)e^x + 2$
5. Determine the values of x for which the following functions are increasing and for which they are decreasing.
   (i) $f(x) = 5x^{3/2} - 3x^{5/2}$ $x > 0$ (ii) $f(x) = \frac{x}{x+3}$ $x \neq -3$
   (ii) $f(x) = x + 1/x$ $x \neq 0$ (iv) $f(x) = x^3 + 1/x^3$ $x \neq 0$

2.4 MAXIMA AND MINIMA OF FUNCTIONS

In this section, we shall study how we can use the derivative to solve problems of finding the maximum and minimum values of a function on an interval.

We begin by looking at the definition of the minimum and the maximum values of a function on an interval.

Definition : Let f be defined on an interval I containing 'c'

1. $f(c)$ is the (absolute) **minimum of *f* on *I*** if $f(c) \leq f(x)$ for all x in I.
2. $f(c)$ is the (absolute) **maximum of *f* on** I if $f(c) \geq f(x)$ for all x in I.

The minimum and maximum of a function on an interval are called the **extreme values** or **extreme,** of the function on the interval.

Remark : A function need not have a minimum or maximum on an interval. For example $f(x) = x$ has neither a maximum nor a minimum on open interval (0,1). Similarly, $f(x) = x^3$ has neither any maximum nor any minimum value in **R**. See figures 15 and 16.

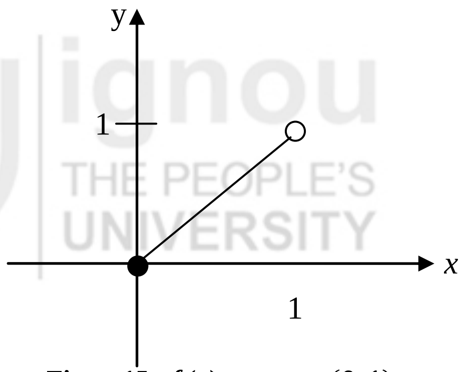

Figure15 : $f(x) = x,\ x \in (0, 1)$

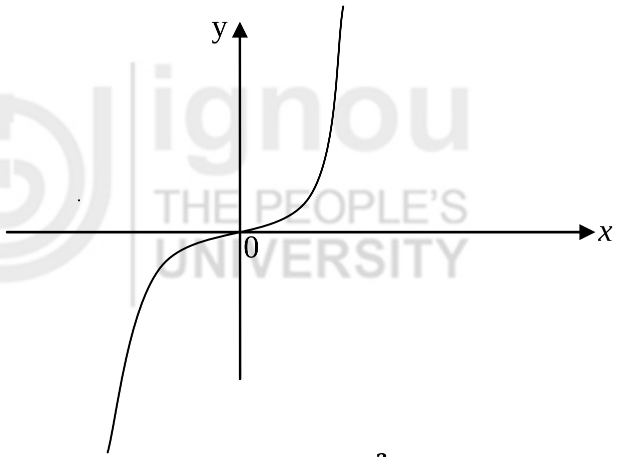

Figure 16 : $f(x)$: $x^3,\ x \in R$

2. If f is a continuous function defined on a closed and bounded interval $[a,b]$, then f has both a minimum and a maximum value on the interval $[a,b]$. This is called the extreme value theorem and its proof is beyond the scope of our course.

   Look at the graph of some function $f(x)$ in figure 17.

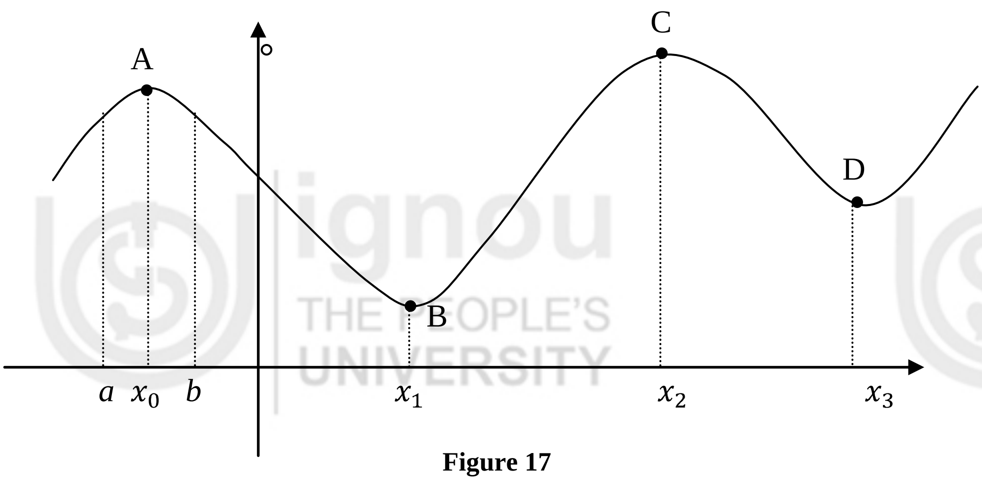

Figure 17

Note that at $x = x_0$, the point A on graph is not an absolute maximum because $f(x_2) > f(x_0)$. But if we consider the interval (a,b), then f has a maximum value at $x = x_0$ in the interval (a,b). Point A is a point of local maximum of f. Similarly f has a local minimum at point B.

Definition : Suppose f is a function defined on an intervals I. f is said to have a local (relative maximum at $c \in I$ for each $x \in I$ for which $c - h < x < c + h, x \neq c$ we have $f(x) > f(c)$.

Definition : Suppose f is a function defined on an interval I. f is said to have a local (relative minimum at $c \in$ I if there is a positive number h such that for each $x \in I$ for which $c - h < x < c + h$, $x \neq c$ we have $f(x) > f(c)$.

Again Fig. 18 suggest that at a relative extreme the derivative is either zero or undefined. We call the x–values at these special points critical numbers.

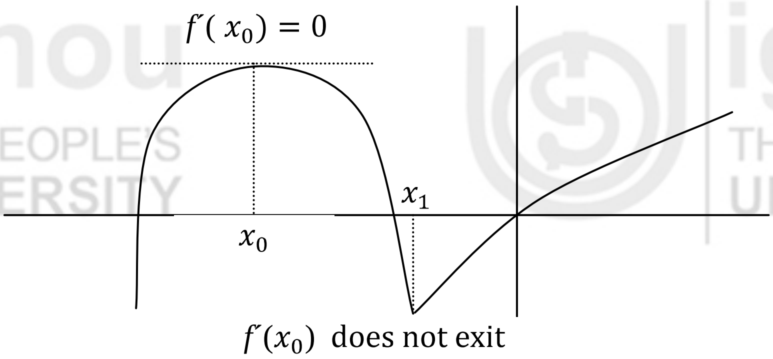

Figure 18

Definition : If f is defined at c, then c is called a critical number if f if $f'(c) = 0$ or f' is not defined at c.

The following theorem which we state without proof tells us that relative extreme can occur only at critical points.

Theorem: If f has a relative minimum or relative maximum at $x = c$, then c is a critical number of f.

If f is a continuous function on interval $[a,b]$, then the absolute extrema of f occur either at a critical number or at the end points a and b. By comparing the values of f at these points we can find the absolute maximum or absolute minimum of f on $[a,b]$.

Example 14 : Find the absolute maximum and minimum of the following functions in the given interval.

(i) $f(x) = x^2$ on $[-3,3]$

(ii) $f(x) = 3x^4 - 4x^3$ on $[-1,2]$

Solution : (i) $f(x) = x^2 \quad x \in [-3,3]$

Differentiating w.r.t. x., we get

$$f'(x) = 2x$$

To obtain critical numbers we set $f'(x) = 0$. This gives $2x = 0$ or $x = 0$ which lies in the interval $(-3,3)$.

Since f' is defined for all x, we conclude that this is the only critical number of f. Let us now evaluate f at the critical number and at the end of points of $[-3,3]$.

$f(-3) = 9$
$f(0) = 0$
$f(3) = 9$

This shows that the absolute maximum of f on $[-3,3]$ is $f(-3) = f(3) = 9$ and the absolute minimum is $f(0) = 0$

(ii) $f(x) = 3x^4 - 4x^3 \quad x \in [-1, 2]$

$f'(x) = 12x^3 - 12x$

To obtain critical numbers, we set $f'(x) = 0$
or $12x^3 - 12x = 0$

which implies $x = 0$ or $x = 1$.

Both these values lie in the interval $(-1, 2)$

Let us now evaluate f at the critical number and at the end points of $[-1,2]$

$f(-1) = 7$
$f(0) = 0$
$f(1) = -1$
$f(2) = 16$

This shows that the absolute maximum 16 of f occurs at $x = 2$ and the absolute minimum -1 occurs at $x = 1$.

First Derivative Test

How do we know whether f has a local maximum or a local minimum at a critical point c ? we shall study two tests to decide whether a critical point c is a point of local maxima or local minima. We begin with the following result which is known **as first derivative test.** This result is stated without any proof.

Theorem : Let c be a critical point for f, and suppose that f is continuous at c and differentiable on some interval I containing c, except possibly at c itself. Then

(i) if f' changes from positive to negative at c, that is, if there exists some $h > 0$ such that $c - h < x < c$, implies $f'(x) > 0$ and $c < x < c + h$ implies $f'(x) < 0$, then f has a local maximum at c.

(ii) if f' changes sign from negative to positive at c, that is, if there exists some $h > 0$ such that $c - h < x < c$ implies $f'(x) < 0$ and $c < x < c + h$ implies $f'(x) > 0$ then f has a local minimum at c.

(iii) if $f'(x) > 0$ or if $f'(x) < 0$ for every x in I except $x = c$ then $f(c)$ is not a local extremum of f.

See the figures 19 and 20 below.

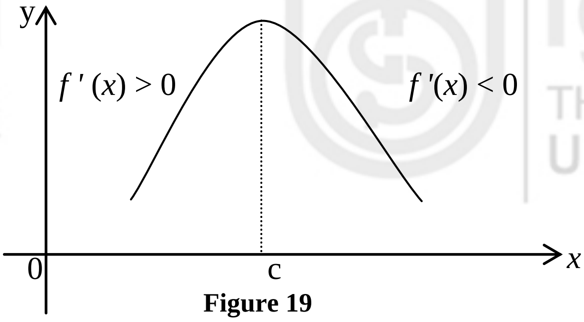

Figure 19

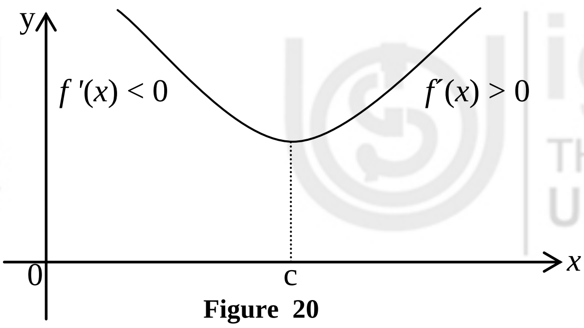

Figure 20

As an illustration of ideas involved, imagine a blind person riding in a car. If that person could feel the car travelling uphill then downhill, he or she would know that the car has passed through a high point of the highway.

Essentially, the sign of derivative $f'(x)$ indicates whether the graph goes uphill or downhill. Therefore, without actually seeing the picture we can deduce the right conclusion in each case.

We summarize the first derivative test for local maxima and minima in the following box.

First Derivative Test for Local Maxima and Minima

Let c be a critical number of f i.e., $f'(c) = 0$

If $f'(x)$ changes sign from positive to negative at c then $f(c)$ is a local maximum. See fig 21.

If $f'(x)$ changes sign from negative to positive at c then $f(c)$ is a local minimum. See fig 22.

Note : $f'(x)$ does not change, sign at c, then $f(c)$ is neither a local maximum nor local minimum.

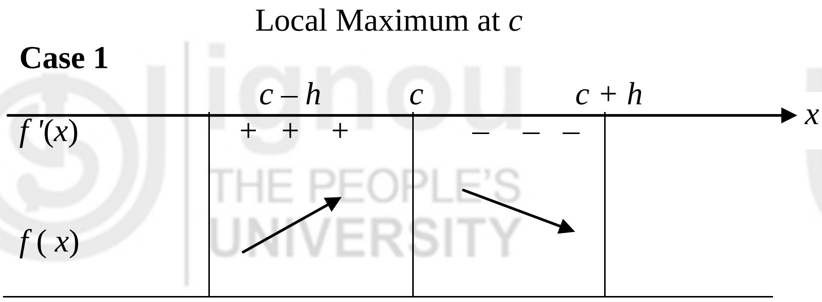

Figure 21

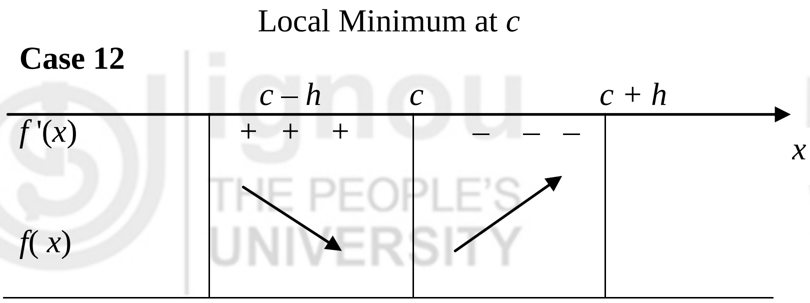

Figure 22

Example 15 : Find the local (relative) extrema of the following functions

(i) $f(x) = 2x^3 + 3x^2 - 12x + 7$ (ii) $f(x) = 1/(x^2 + 2)$

(iii) $f(x) = x\,e^x$ (iv) $f(x) = \frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105$

Solution

(i) f is continuous and differentiable on **R**, the set of real numbers. Therefore, the only critical values of f will be the solutions of the equation $f'(x) = 0$.
Now, $f'(x) = 6x^2 + 6x - 12 = 6(x^2 + x - 2) = 6(x+2)(x-1)$

Setting $f'(x) = 0$ we obtain $x = -2, 1$

Thus, $x = -2$ and $x = 1$ are the only critical numbers of f. Figure 23 shows the sign of derivative f' in three intervals.

From Figure 23 it is clear that if $x < -2$, $f'(x) > 0$; if $-2 < x < 1$, $f'(x) < 0$ and if $x > 1$, $f'(x) > 0$.

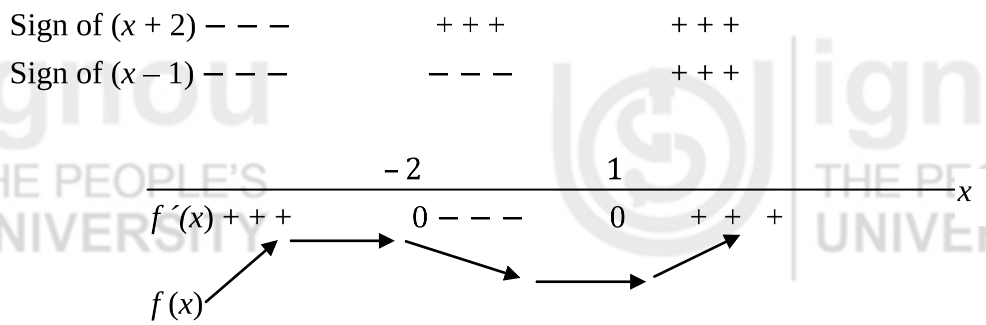

Figure 23

Using the first derivative test we conclude that $f(x)$ has a local maximum at $x = – 2$ and $f(x)$ has local minimum at $x = 1$.

Now, $f(–2) – 2 = 2(-2)^3 + 3(-2^2) –12(–2) + 7 = –16 + 12 + 24 + 7 = 27$ is the value of local maximum at $x = – 2$ and $f(1) = 2 + 3 – 12 + 7 = 0$ is the value of local minimum at $x = 1$.

(ii) Since $x^2 + 2$ is a polynomial and $x^2+2 \neq 0 \;\forall\, x \in \boldsymbol{R}$ $f(x) = \frac{1}{x^2+2}$ is continuous and differentiable on **R**, the set of real numbers. Therefore, the only critical values of f will be the solutions of the equation $f\,'(x) = 0$

$$\text{Now, } f\,'(x) = \frac{-2x}{(x^2+2)^2}$$

Setting $f\,'(x) = 0$ we obtain $x = 0$. Thus, $x = 0$ is the only critical number of f. Figure 24 shows the sign of derivative in two intervals.

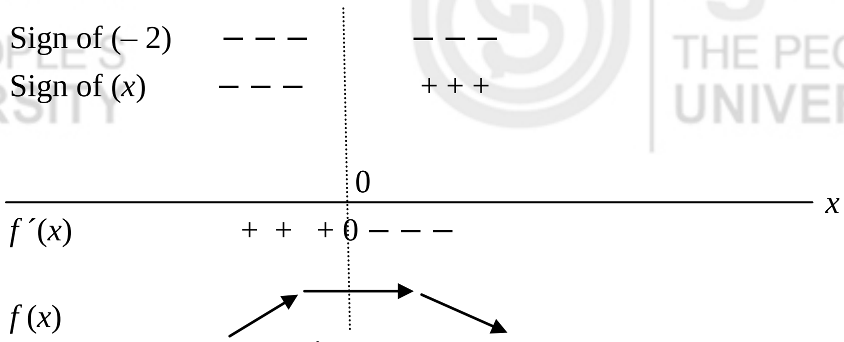

Figure 24

From Figure 24 it is clear that $f\,'(x) > 0$ if $x < 0$ and $f\,'(x) < 0$ if $x > 0$

Using the first derivative test, we conclude that $f(x)$ has a local maximum at $x = 0$.

Now since $f(0) = \dfrac{1}{0^2+2} = \dfrac{1}{2}$ the value of local maximum at $x = 0$ is 1/2.

(iii) Since x and e^x are continuous and differentaible on $\boldsymbol{R}$, $f(x) = xe^x$ is continuous and differentiable on **R.**

Therefore, the only critical values of f will be solutions of $f\,'(x) = 0$.
Now, $f\,'(x) = x\,e^x + (1)e^x = (x+1)e^x$

Since $e^x > 0 \ \forall \ x \in R$ $f'(x) = 0$ gives $x = -1$. Thus, $x = -1$ is the only critical number of f.

Figure 25 shows the sign of derivative f' in two intervals

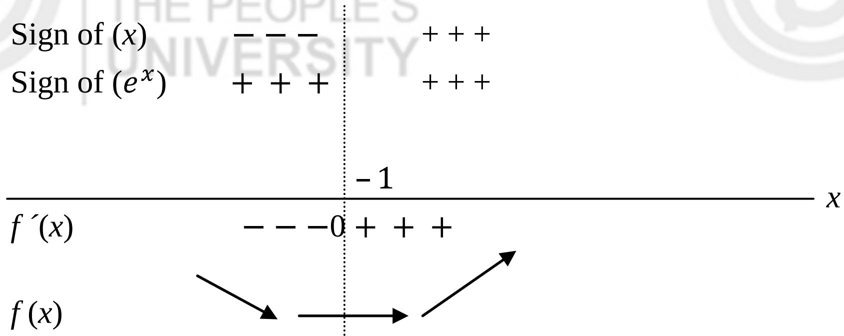

Figure 25

From Figure 25 it is clear that $f'(x) < 0$ if $x < -1$ and $f'(x) > 0$ if $x > -1$

Using the first derivative test we conclude that $f(x)$ has a local minimum at $x = -1$ and the value of local minimum is $f(-1) = (-1)\, e^{-1} = -1/e$.

(iv) Since f is a polynomial function, f is continuous and differentiable on **R.** Therefore, the only critical numbers of f are the solutions of the equation $f'(x) = 0$.

We have
$$f'(x) = -3x^3 - 24x^2 - 54x$$
$$= -3(x^2 + 8x + 15)x$$
$$= -3(x+5)(x+3)x$$

Setting $f'(x) = 0$, we obtain $x = -5, -3, 0$ Thus, $-5, -3$ and 0 are the only critical numbers of f.

Figure 26 shows the sign of derivative f' in four intervals.

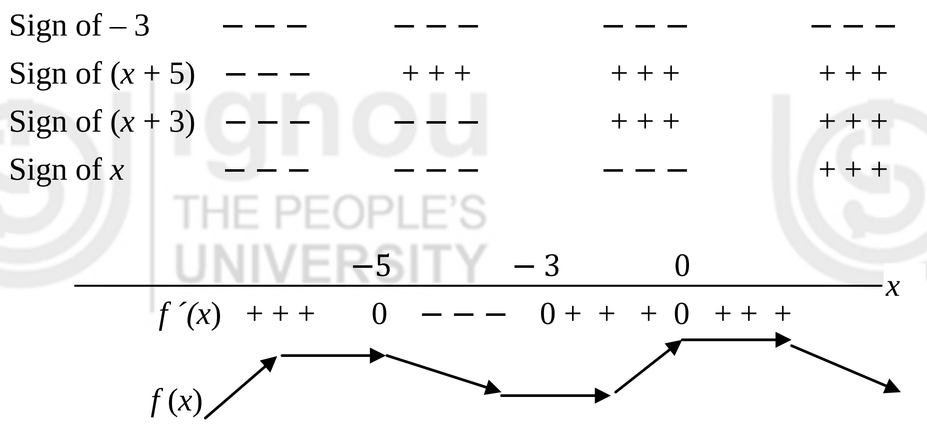

Figure 26

It is clear from figure 26 that $f'(x) > 0$ for $x < -5$, $f'(x) < 0$ for $-5 < x < -3$, $f'(x) > 0$ for $-3 < x < 0$ and $f'(x) < 0$ for $x > 0$

Using the first derivative test we get that $f(x)$ has a local maximum at $x = -5$, a local minimum at $x = -3$ and a local maximum at $x - 0$.

Values of local maximum at $x = -5$ is $f(-5) = 73.75$ and value of local minimum at $x = -3$ is $f(-3) = 57.75$ and the value of local maximum at 0 is $f(0) = 105$.

Second Derivative Test

The first derivative test is very useful for finding the local maxima and local minima of a function. But it is slightly cumbersome to apply as we have to determine the sign of f' around the point under consideration. However, we can avoid determining the sign of derivative f' around the point under consideration, say c, if we know the sign of second derivative f at point c. We shall call it as the second derivative test.

Theorem : (Second Derivative Test)

Let $f(x)$ be a differentiable function on I and let $c \in I$. Let $f'(x)$ be continuous at c. Then

1. c is a point of local maximum if both $f'(c) = 0$ and $f''(c) < 0$.
2. c is a point of local minimum if both $f'(c) = 0$ and $f''(c) > 0$.

Remark : If $f'(c) = 0$ and $f''(c) = 0$, then the second derivative test fails. In this case, we use the first derivative test to determine whether c is a point of local maximum or a point of a local minimum.

We summarize the second – derivative test for local maxima and minima in the following table.

Second Derivative Test for Local Maxima and Minima

$f'(c)$	$f''(c)$	$f(c)$
0	+	Local Minimum
0	–	Local Maximum
0	0	Test Fails

We shall adopt the following guidelines to determine local maxima and minima.

Guidelines to find Local Maxima and Local Minima

The function f is assumed to posses the second derivative on the interval *I*.

Step 1 : Find $f'(x)$ and set it equal to 0.

Step 2 : Solve $f'(x) = 0$ to obtain the critical numbers of *f*.

Let the solution of this equation be $\alpha, \beta, \gamma \ldots\ldots\ldots$

We shall consider only those values of *x* which lie in *I* and which are not end points of *I*.

Step 3 : Evaluate $f''(\alpha)$

If $f''(\alpha) < 0$, $f(x)$ has a local maximum at $x = \alpha$ and its value if $f(\alpha)$

If $f''(\alpha) > 0$, $f(x)$ has a local minimum at $x = \alpha$ and its value if $f(\alpha)$

If $f''(\alpha) = 0$, apply the first derivative test.

Step 4 : If the list of values in Step 2 is not exhausted, repeat step 3, with that value.

Example 16 : Find the points of local maxima and minima, if any, of each of the following functions. Find also the local maximum values and local minimum values.

(i) $f(x) = x^3 – 6x^2 + 9x + 1$, $x \in \boldsymbol{R}$

(ii) $f(x) = \frac{1}{6}x^6 – 4x^5 + 25x^4$, $x \in \boldsymbol{R}$

(iii) $f(x) = x^3 – 2ax^2 + a^2 x \ (a > 0)$, $x \in \boldsymbol{R}$

Solution : $f'(x) = x^3 – 6x^2 + 9x + 1$

Thus, $f'(x) = 3x^2 - 12x + 9 = 3(x^2 – 4x + 3) = 3(x – 1)(x – 3)$.

To obtain critical number of *f*, we set $f'(x) = 0$ this yields $x = 1, 3$.

Therefore, the critical number of *f* are $x = 1, 3$.

Now $f''(x) = 6x – 12 = 6(x – 2)$

We have $f''(1) = 6(1 – 2) = – 6 < 0$ and $f''(3) = 6(3 – 2) = 6 > 0$.

Using the second derivative test, we see that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 3$. The value of local maximum at $x = 1$ is $f(1) = 1 – 6 + 9 + 1 = 5$ and the value of local minimum at $x = 3$ is $f(3) = 3^3 – 6(3^2) + 9(3) + 1 = 27 – 54 + 27 + 1 = 1$.

(ii) We have $f'(x) = \dfrac{1}{6}x^6 - 4x^5 + 25x^4$

Thus, $f'(x) = x^5 - 20x^4 + 100x^3 = x^3(x^2 - 20x + 100) = x^3(x - 10)^2$

As $f'(x)$ is defined for every value of *x*, the critical number *f* are solutions of $f'(x) = 0$. Setting $f'(x) = 0$, we get $x = 0$ *or* $x = 10$.

Now $f''(x) = 3x^2(x – 10)^2 + x^3(2)(x – 10)$

$$= x^2(x - 10)[3(x - 10) + 2x] = 5x^2(x - 10)(x - 6)$$

We have $f'(0) = 0$ and $f'(10) = 0$. Therefore we cannot use the second derivative test to decide about the local maxima and minima. We, therefore use the first derivative test.

Figure 27 shows the sign of derivative f' in three intervals.

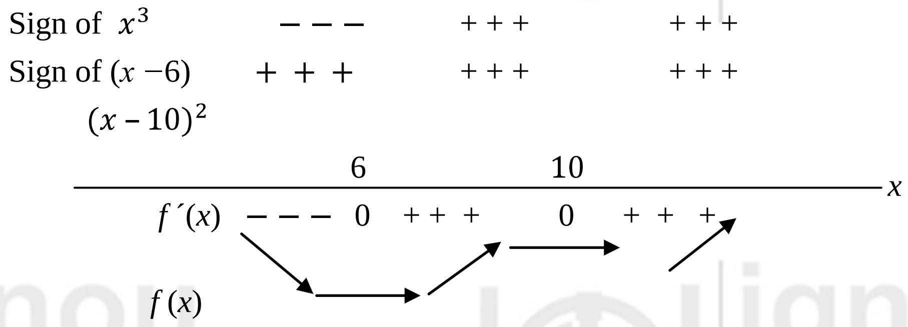

Figure 27

From figure 27 it is clear that $f'(x) < 0$ for $x < 0$, $f'(x) > 0$ for $0 < x < 10$ and $f'(x) > 0$ for $x > 10$

Thus, $f(x)$ has a local minimum at $x = 0$ and its value is $f(0) = 0$. But $f(x)$ has neither a local maximum nor a local minimum at $x = 10$.

(iii) We have $f(x) = x^3 - 2ax^2 + a^2 x \ (a > 0)$,

Thus, $f(x) = 3x^2 - 4ax + a^2 = (3x - a)(x - a)$

As $f'(x)$ is defined for each $x \in \boldsymbol{R}$, to obtain critical number of f we set $f'(x) = 0$. This yields $x = a/3$ or $x = a$. Therefore, the critical numbers of f and $a/3$ and a. Now, $f''(a) = 6x - 4a$.

We have $f''\left(\frac{a}{3}\right) = 6\left(\frac{a}{3}\right) - 4a = 2a - 4a = -2a < 0$ and $f(a) = 6a - 4a = 2a > 0$.

Using the second derivative test, we see that $f(x)$ has a local maximum at $x = a/3$ and a local minimum at $x = a$. The value of local maximum at

$$x = \frac{a}{3} \text{ is } f\left(\frac{a}{3}\right) = \left(\frac{a}{3}\right)^3 - 2a\left(\frac{a}{3}\right)^2 + a^2\left(\frac{a}{3}\right) = \frac{4}{27}a^3$$

and the value of local minimum at $x = a$ is $f(a) = a^3 - 2a.\ a^2 + a.a = 0$

Example 17 : Show that $f(x) = x^2 \ln\left(\frac{1}{x}\right)$ has a local maximum at $x = 1/\sqrt{e}$.

Solution : We have $f(x) = x^2 \ln\left(\frac{1}{x}\right) = -x^2 \ln x \quad x > 0$.

Thus, $f'(x) = -2x \ln x - x^2 \left(\frac{1}{x}\right) = -2x \ln x - x$.

Note that $f'(x)$ is defined for each $x > 0$. Since, we have to show that $f(x)$ has a local maximum at $x = 1/\sqrt{e}$, it is sufficient to show that $f'(1/\sqrt{e}) < 0$ and

$f'(1/\sqrt{e}) < 0$. We have

$$f'\left(\frac{1}{\sqrt{e}}\right) = 2\left(\frac{1}{\sqrt{e}}\right) \ln\left(e^{-\frac{1}{2}}\right) - \frac{1}{\sqrt{e}} = \frac{-2}{\sqrt{e}}\left(-\frac{1}{2}\right) \ln(e) - \frac{1}{\sqrt{e}} = \frac{1}{\sqrt{e}} - \frac{1}{\sqrt{e}} = 0.$$

Now, $f'(x) = -2(1) \ln x - 2x(1/x) - 1 = -2 \ln x - 3$

$$\Rightarrow f'\left(\frac{1}{\sqrt{e}}\right) = -2\ln\left(\frac{1}{\sqrt{e}}\right) - 3 = -2\ln\left(e^{-\frac{1}{2}}\right) - 3 = \ln e - 3 = 1 - 3 = -2 < 0.$$

Thus, $f(x)$ has a local maximum at $x = 1/\sqrt{e}$ and its value is

$$f'\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e} \ln\left(\frac{1}{\sqrt{e}}\right) = \frac{1}{e} \ln\left(e^{-\frac{1}{2}}\right) = -\frac{1}{2e}.$$

Check Your Progress – 3

1. Find the absolute maximum and minimum of the following functions in the given intervals.
   (i) $f(x) = 4x^2 - 7x + 3$ on $[-2, 3]$
   (ii) $f(x) = \frac{x^3}{x+2}$ on $[-1,1]$
2. Using first derivative test find the local maxima and minima of the following functions.
   (i) $f(x) = x^3 - 12x$
   (ii) $f(x) - \frac{x}{2} + \frac{2}{x}$, $x > 0$

3. Use second derivative test to find the local maxima and minima of the following functions.
   (i) $f(x) = x^3 - 2x^2 + x + 1$ $x \in \mathbf{R}$
   (ii) $f(x) = x + 2\sqrt{1-x}$ $x \leq 1$

2.5 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. We are given the rate of change of the side of the square and require the rate of change of the perimeter.

   Let s cm be the side of the square and P cm be the perimeter of the square at time t. Thus
   $$P = 4s$$
   Differentiating both the sides with respect to t, we get

   $$\frac{dP}{dt} = 4\frac{ds}{dt}$$

   But $\frac{ds}{dt} = 0.2$ therefore, $\frac{dP}{dt} = 4(0.2) = 0.8$

   Thus, the perimeter of the square is increasing at the rate of 0.8 cm/s.

2. We are given that the rate of change of radius and require the rate of change of the circumference.

   Let r cm be the radius of the circle and C cm be circumference of the circle at time t.

   Thus, $C = 2\pi r$

   Differentiating both sides with respect to t, we get

$$\frac{dC}{dt} = 2\pi \frac{dr}{dt}. \text{ But } \frac{dr}{dt} = 0.7, \text{ therefore, } \frac{dC}{dt} = 2\pi(0.7) = 1.4\pi.$$

   Thus, the circumference of the circle is increasing at the rate of $1.4\,\pi\, cm/s$.

3. Let x be the distance of the man from the base of the light post, and y be the length of the shadow (see figure 28)

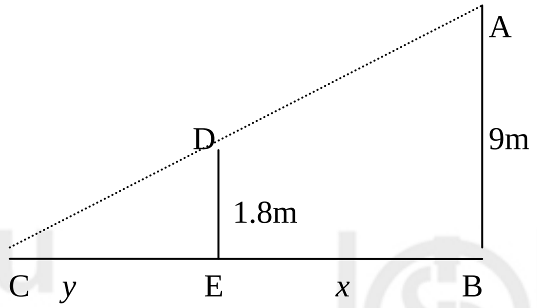

Figure 28

Δ ABC is simliar to Δ DEC. Hence, $\frac{AB}{DE} = \frac{BC}{CE}$ which implies $\frac{9}{1.8} = \frac{x+y}{x}$

$\Rightarrow 5y = x + y$ or $4y = x$ or $y = x/4$.

Differentiating both the sides with respect to t, we get $\frac{dy}{dt} = \frac{1}{4}\frac{dx}{dt}$.

But we are given that $\frac{dx}{dt} = 2$. Thus, $\frac{dy}{dt} = \frac{1}{4}(2) = 0.5$

   This shows that the shadow is lengthening at the rate of 0.5 m/s.

4. The fluid in the funnel forms a cone with radius r, height h and volume V (figure 29). Recall that the volume of the cone is given by $\frac{1}{3}\pi r^2 h$.

   Therefore, $V = \frac{1}{3}\pi r^2 h$

Note that ΔAOB and ΔADC are similar.

Therefore,

$$\frac{AO}{OB} = \frac{AD}{DC}$$

$$or\ \frac{20}{4} = \frac{h}{r}$$

$$or\ r = \frac{1}{5}\ h$$

Thus, $V = \frac{1}{3}\pi\left(\frac{1}{5}h\right)^2 = \frac{1}{75}\pi h^3$

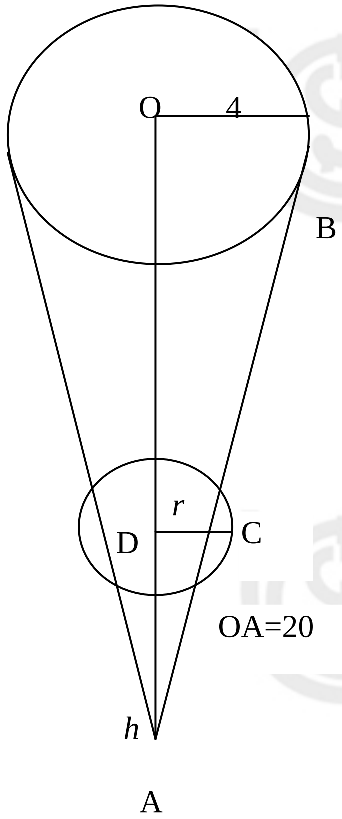

Figure 29

Differentiating both the sides with respect to *t*, we get

$$\frac{dV}{dt} = \frac{1}{75}\ \pi(3h^2)\ \frac{dh}{dt} = \frac{\pi h^2}{25}\frac{dh}{dt}$$

We are given that $\frac{dV}{dt} = -12$ (Negative sign is due to decrease in volume)

Therefore, $-12 = \frac{\pi h^2}{25}\frac{dh}{dt}$ Or $\frac{dh}{dt} = \frac{-300}{\pi h^2}$

Thus, when $h = -5$, $\quad \frac{dh}{dt} = \frac{-300}{\pi 25} = \frac{-12}{\pi}$

This shows, that the fluid is dropping at the rate of $12/\pi$cm/s.

Check Your Progress 2

1. First note that the domain of *f* consists of non–negative real number. Now, let $0 \le x_1 < x_2$.

   Since $x_1 < x_2$ and $x_1 \ge 0$, it follows that
   $x_1^2 \le x_1 x_2$ ....(1)

   Again, $x_1 < x_2$ and $x_2 > 0$, it follows that
   $x_1 x_2 < x_2^2$ ....(2)

   From (1) and (2), we get

   $$x_1^2 < x_1 x_2 < x_2^2$$

   $$i.e.,\quad x_1^2 < x_2^2$$

   which means $f(x_1) < f(x_2)$

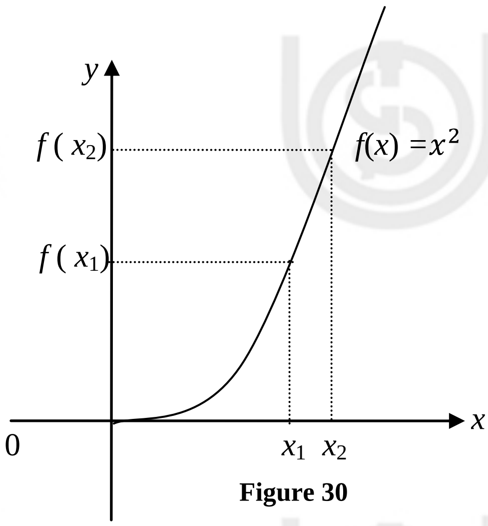

Figure 30

Hence, *f* is an increasing function on the interval $(0, \infty)$. See also figure 30

2. (a) Let $a > 0$

Suppose $x_1, x_1 \in \mathbf{R}$ and $x_1 > x_2$ As $a > 0$,

we get $a\,x_1 > a\,x_2$

$\Rightarrow a\,x_1 + b > a\,x_2 + b \;\Rightarrow\; f(x_1) > f(x_2)$

Thus, f increase on $\mathbf{R}$. See figure 31.

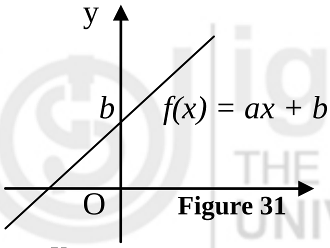

Figure 31

(b) Next, let $a < 0$.

Suppose $x_1, x_2 \in \mathbf{R}$ *and* $x_1 > x_2$. As $a < 0$.

we get $a\,x_1 < ax_2 \quad \Rightarrow$

$a\,x_1 + b < a\,x_2 + b \Rightarrow f(x_1) < f(x_2)$.

Thus, f decreases on $\mathbf{R}$. See figure 32.

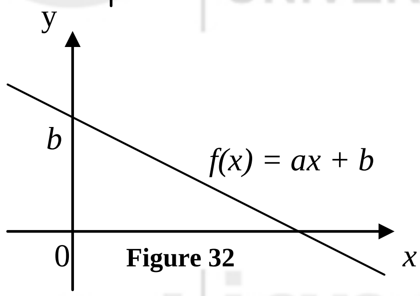

Figure 32

3. We have $f'(x) = 6\,x^2 - 30x + 36 = 6(x^2 - 5x + 6) = 6(x-2)(x-3)$

Sign of $(x-2)$ $- - -$ $\quad + + +$ $\quad + + +$

Sign of $(x-3)$ $- - -$ $\quad - - -$ $\quad + + +$

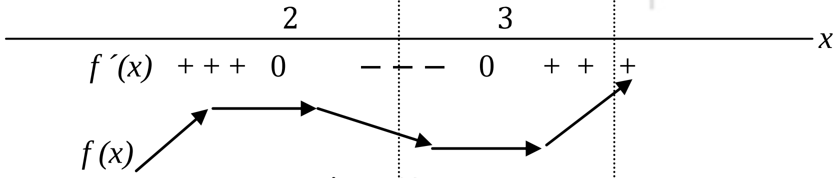

Figure 33

See figure 33 and note that

If $x < 2$, then $x - 2 < 0$, $x - 3 < 0$ and therefore, $f'(x) > 0$,

If $2 < x < 3$, then $x - 2 > 0$, $x - 3 < 0$ and therefore, $f'(x) < 0$,

If $x > 3$, then $x - 2 > 0$, $x - 3 > 0$ and therefore, $f'(x) > 0$.

Thus, $f(x)$ increases on $(-\infty, 2] \cup [3, \infty)$ and decreases on $[2,3]$.

4. (i) We have $f'(x) = 8\,x^7 + 12\,x = 4x\,(2\,x^6 + 3)$.

Since $2x^6 + 3 > 0\ \forall\ x \in R$ $f'(x) < 0$ and $x < 0$ and $f'(x) > 0$ for $x > 0$.

Thus, $f(x)$ decreases for $x \le 0$ and increases for $x \ge 0$.

(ii) We have $f'(x) = \dfrac{(1 + x^2)(1) - x(2x)}{(1 + x^2)^2} = \dfrac{1 - x^2}{(1 + x^2)^2} = \dfrac{-(x+1)(x-1)}{(1 + x^2)^2}$

Sign of (-1) $- - -$ $\quad - - -$ $\quad - - -$

Sign of $(x-1)$ $- - -$ $\quad + + +$ $\quad + + +$

Sign of $(x-1)$ $- - -$ $\quad - - -$ $\quad + + +$

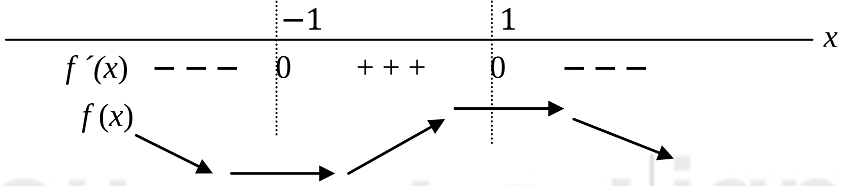

Figure 34

Referring to Figure 33, we get $f'(x) < 0$ if $x < -1$ or $x > 1$
and $f'(x) > 0$ if $-1 < x < 1$
Thus, $f'(x) > 0$ if $x < -1$ or $x > 1$ and $f'(x) < 0$ if $-1 < x < 1$.
Hence, $f(x)$ decreases on $(-\infty, -1] \cup [1, \infty)$ and increases on $[-1,1]$.

(iii) We have $f'(x) = 4x^3 - 4 = 4(x^3 - 1) = 4(x-1)(x^2 + x + 1)$

[using $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$]

$$= 4(x-1)[x^2 + 2\left(\frac{1}{2}\right)x + 1] = 4(x-1)[(x + \tfrac{1}{2})^2 + \tfrac{3}{4}]$$

Since $(x + 1/2)^2 + 3/4 > 0 \; \forall \; x \in \mathbf{R}$, we get $f'(x) < 0$ if $x < 1$
and $f'(x) > 0$ if $x > 1$.
Thus, $f'(x)$ decrease on $(-\infty, 1]$ and increase on $[1, \infty)$

(iv) We have $f'(x) = (x-1)e^x + (1)e^x = (x - 1 + 1)e^x = x\,e^x$.

As $e^x > 0 \; \forall \; x \in \boldsymbol{R}$ $f'(x) > 0$ for $x > 0$ and $f'(x) < 0$ for $x < 0$.

Thus, $f(x)$ increases for $x \geq 0$ and decreases for $x \leq 0$

5. (i) We have, $x > 0$, $f'(x) = \frac{15}{2}x^{1/2} - \frac{15}{2}x^{3/2} = \frac{15}{2}\sqrt{x}\,(1 - x)$.

As $\sqrt{x} > 0$, for $x > 0$ we get $f'(x) > 0$ if $0 < x < 1$ and $f'(x) < 0$ if $x > 1$.

Thus, $f'(x) > 0$ if $0 < x < 1$ and $f'(x) < 0$ if $x > 1$.

Hence, $f(x)$ increases for $0 < x \leq 1$ and decreases for $x \geq 1$.

(ii) We have $x \neq -3$ $f'(x) = \dfrac{(x+3)(1) - x(1)}{(x+3)^2} = \dfrac{3}{(x+3)^2}$

$\Rightarrow f'(x) > 0$ for $x \neq -3$. Thus, $f(x)$ increases on $(-\infty, -3) \cup (-3, -\infty)$

(iii) We have for $x \neq 0$ $f'(x) = 1 - \dfrac{1}{x^2} = \dfrac{x^2 - 1}{x^2} = \dfrac{(x+1)(x-1)}{x^2}$

Since $x^2 > 0$ for $x \neq 0$ $f'(x) > 0$ if $(x+1)(x-1) > 0$ $x \neq 0$
and $f'(x) < 0$ if $(x+1)(x-1) < 0$, $x \neq 0$.

Note that $f'(x) > 0$ if $x < -1$ or $x > 1$ and $f'(x) < 0$ if $-1 < x < 0$ or $0 < x < 1$.

Thus, $f(x)$ increases on $(-\infty, -1] \cup (1, \infty)$ and decreases on
$[-1, 0] \cup (0,1]$

(iv) We have $f'(x) = 3x^2 - \dfrac{3}{x^4} = \dfrac{3(x^6 - 1)}{x^4} = \dfrac{3(x^2 - 1)(x^4 + x^2 + 1)}{x^4}$

[$\because a^3 - b^3 = (a-b)(a^2 + ab + b^2)$]

Since $x^4+x^2+1>0$ and $x^4>0$ $for\ x \neq 0,$ we get $f'(x)>0$ $if\ x^2-1>0$ and $f'(x)<0$ $if\ x^2-1<0$. As in (iii), we get *f(x)* increases on $(-\infty,-1] \cup [1,\infty)$ and decreases on $[-1, 0] \cup (0,1]$.

Check Your Progress 3

1. (i) $f(x) = 4x^2 - 7x + 3,\ \ x \in [-2,3]$

Differentiating with respect to *x*, we get

$f'(x) = 8x - 7$

To obtain critical numbers, we set $f'(x) = 0$. This gives $8x - 7 = 0$ or $x = 7/8$, which lies in the interval (–2,3). Since f' is defined for all *x*, we conclude that this is the only critical number of *f*. We now evaluate *f* at the critical number and at the end points of [–2, 3].

We have

$f(-2) = 4(-2)^2 + 7(-2) + 3 = 33$

$$f\left(\frac{7}{8}\right) = f\left(\frac{7}{8}\right)^2 - 7\left(\frac{7}{8}\right) + 3 = -\frac{1}{16}$$

$f(3) = 4(3)^2 - 7(3) + 3 = 18$

This shows that the absolute maximum of *f* on [–2,3] is $f(-2) = 33$ and the absolute minimum is $f(7/8) = -\frac{1}{16}$.

(ii) $f(x) = \dfrac{x^3}{x+2},\ \ x \in [-1,1]$

Differentiating w.r.t. *x*, we get

$$f(x) = \frac{2x^2(x+3)}{(x+2)^2}$$

Note that $f'(x)$ is not defined for $x = -2$. However, it does not lie in the interval (–1,1).

To obtain critical number of *f* we set $f'(x) = 0$. This gives $2x^2(x+3) = 0$ which implies $x = 0$ or $x = -3$ Since –3 does not lie in (–1, 1), 0 is the only critical number of *f*. We therefore evaluate *f* at –1, 0 and 1. Now

$f(-1) = -1$

$f(0) = 0$

and $f(1) = 1/3$

This shows that *f* has the absolute maximum at $x = 1$ and the absolute minimum at $x = -1$.

2. (i) $f(x) = x^3 - 12x$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 2(x-2)(x+2)$

Setting $f'(x) = 0$, we obtain $x = 2, -2$ Thus, $x = -2$, and $x = 2$ are the only critical numbers of f. Fig. 35 shows the sign of derivative f' in three intervals.

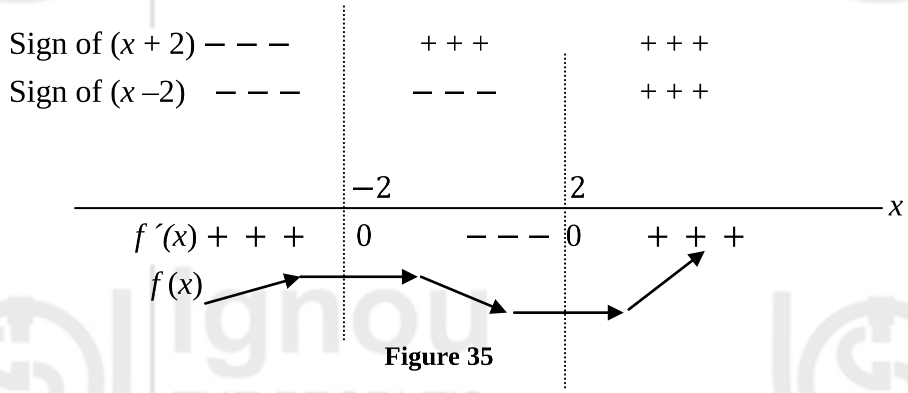

Figure 35

From figure 35 it is clear that if $x < -2$, $f'(x) > 0$; if $-2 < x < 2$, $f'(x) < 0$ and if $x > 2$, $f'(x) > 0$.

Using the first derivative test, we conclude that

$f(x)$ has a local maximum at $x = -2$ and a local minimum *at* $x = 2$.

Now, $f(-2) = (-2)^3 - 12(-2) = -8 + 24 = 16$ is the value of local maximum at $x = -2$ and $f(2) = 2^3 - 12(2) = 8 - 24 = -16$ is the value of the local minimum at $x = 2$.

(ii) We have $f(x) = \dfrac{x}{2} + \dfrac{2}{x} = \dfrac{1}{2}x + 2x^{-1}, x > 0.$

$$\text{Thus, } f'(x) = \frac{1}{2} + 2(-1)x^{-2} = \frac{1}{2} - \frac{2}{x^2} = \frac{x^2 - 4}{2x^2} = \frac{(x+2)(x-2)}{2x^2}$$

Note that $f'(x)$ is defined for each $x > 0$, therefore to obtain the critical numbers of f, we set $f'(x) = 0$. This gives us $(x + 2)(x - 2) = 0$ or $x = -2, 2$. As domain of f is $\{x | x > 0\}$, the only critical number of f lying in the domain of f is 2.

Also, since $2x^2 > 0$ and $x + 2 > 0 \; \forall \; x > 0$, the sign of $f'(x)$ is determined by the sign of the factor $x - 2$.

Figure 36 give the sign of $f'(x)$ in two intervals.

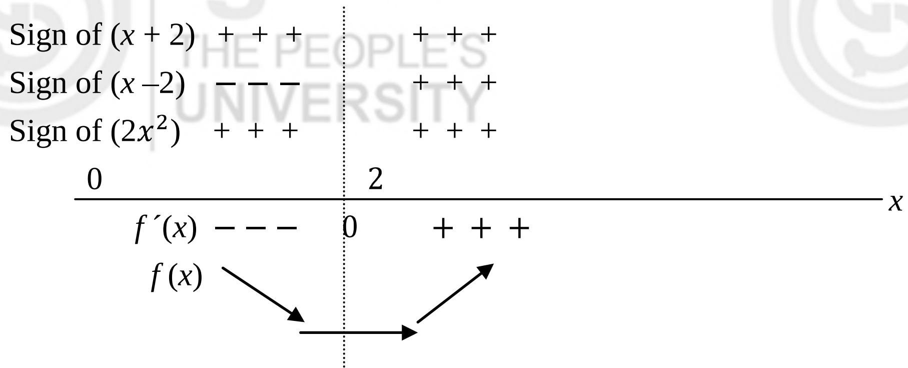

Figure 36

From figure 36 it is clear that $f'(x) < 0$ for $0 < x < 2$ and $f'(x) > 0$ for $x > 2$. Thus, $f(x)$ has a local minimum at $x = 2$ and its value is

$$f(2) = \frac{2}{2} + \frac{2}{2} = 1 + 1 = 2.$$

3. (i) $f(x) = x^2 - 2x^2 + x + 1, \quad x \in R$

Differentiating w.r.t. x, we get

$f'(x) = 3x^2 - 4x + 1$
$\quad = (3x - 1)(x - 1)$

For obtaining critical numbers, we set $f'(x) = 0$

or $(3x - 1)(x - 1) = 0$

or $x = 1/3$ and $x = 1$

So, $x = 1$ and $x = 1/3$ are the only critical number of f.

Now $f(x) = 6x - 4$

We have

$f''(1) = 6(1) - 4 = 2 > 0$

and $f''(1/3) = 6(1/3) - 4 = -2 < 0$
Using the second derivative test we see that $f(x)$ has a local maximum at $x = 1/3$ and a local minimum at $x = 1$.

Now $f''(1/3) = 31/27$ and $f(1) = 1$

Hence, f has a local maximum 31/27at $x = 1/3$ and a local minimum 1 at $x = 1$.

(ii) We have $f(x) = x + 2\sqrt{1-x} = x + (1-x)^{1/2} \quad x \leq 1$

Thus, $f'(x) = 1 - (1-x)^{-1/2}\ (-1) = 1 - \frac{1}{\sqrt{1-x}}$
Note that $f'(x)$ is defined for each $x < 1$

To obtain the critical numbers of f, we set $f'(x) = 0$ but $f'(x) = 0$ implies $1 - \dfrac{1}{\sqrt{1-x}}$ or $\sqrt{1-x} = 1$ or $1 - x = 1$ or $x = 0$.

This lies in the domain of f. Therefore, the only critical of f is $x = 0$.

Now, $\quad f(x) = 0 - \left(-\dfrac{1}{2}\right)(1-x)^{-3/2}(-1) = \dfrac{-1}{(1-x)^{-3/2}}$

We have $f(0) = \dfrac{-1}{2(1-0)^{-3/2}} = -\dfrac{1}{2} < 0$
Using the second derivative test we see that $f(x)$ has a local maximum at $x = 0$. The value of local maximum at $x = 0$ $f(0) = 0 + \sqrt{1-0} = 1$

2.6 SUMMARY

The unit is, as suggested by the title, on applications of differential calculus. In **section 2.2**, the concept of 'rate of change' of a derivable variable/quantity is introduced and illustrated with a number of examples. In **section 2.3**, concepts of 'increasing function' and 'decreasing function' are discussed and explained with a number of examples. In **section 2.4**, methods for finding out (local) maxima and minima, are discussed and explained with examples.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 2.5**.