UNIT 4 APPLICATIONS OF INTEGRAL CALCULUS

Structure

4.0 INTRODUCTION

In the previous unit, we introduced the concept of indefinite integral (anti-derivative). We now look at a new problem – that of finding the area of a plane region. At first glance these two ideas seem, to be unrelated, but we shall see in this unit that ideas are closely related by an important theorem called the Fundamental Theorem of Calculus. We shall also learn how to use integration to find the length of curves.

4.1 OBJECTIVES

After studying this unit you should be able to :

- evaluate the definite integral $\int_a^b f(x)dx$;
- use definite integral to find area under a curve and area between two curves; and
- use definite integral to find length of a curve.

4.2 DEFINITE INTEGRAL

Suppose f is a continuous function defined on the interval $[a,b]$, and let F be an antiderivative of f, then we write

$$\int_a^b f(x)dx = \left[\mathrm{F}(x)\right]_a^b = \mathrm{F}(b) - \mathrm{F}(a)$$

For instance

1. $\int_1^2 xdx = \left[\frac{x^2}{1}\right]_1^2 = \frac{1}{2}(2^2 - 1^1) = \frac{3}{2}$

2. $\int_2^3 \frac{1}{x} dx \quad [\log|x|]_2^3 = \log 3 - \log 2 = \log(3/2)$

3. $\int_0^2 e^x dx = [e^x]_0^2 = e^2 - e^0 = e^2 - 1$

4. $\int_0^1 \frac{dx}{x+1} = [\log|x+1|]_0^1 = \log_2 - \log_1 = \log_2$

Some Properties of Definite Integral

We list these properties without proof.

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = - \int_b^a f(x)dx$

3. $\int_a^b f(t)dt = \int_a^b f(u)du = \int_a^b f(x)dx$

4. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

5. $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

6. If f is an odd function, that is $f(-x) = -f(x) \;\; \forall\, x \in [-a,a]$, then

   $\int_{-a}^a f(x)dx = 0$

   For instance,

   $\int_{-1}^1 x^3 dx = 0$

7. If f is an even function, that is, $f(-x) = f(x) \;\; \forall\, x \in [-a, a]$, then

   $\int_{-a}^a f(x)dx = 2 \int_0^a f(x)dx$

8. $\int_0^{2a} f(x)dx = \int_0^a [f(x) + f(2a-x)]dx$

9. $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$

Fundamental Theorem of Calculus

Let f be a continuous function defined on a closed and bounded interval $[a,b]$.
Let

$$\mathrm{A}(x) = \int_0^x f(t)dt,$$

Then $\mathrm{A}'(x) = f(x)$ $\forall$ x € $[a,b]$.

Illustration

Let $f(x) = x$ for $1 \le x \le 3$.

Let $\mathrm{A}(x) = \int_1^x f(t)dt = \int_1^x t dt = \left[\frac{1}{2}t^2\right]_1^x = \frac{1}{2}(x^2 - 1)$

$\Rightarrow \mathrm{A}'(x) = \frac{1}{2}(2x) = x = f(x)$ for $1 \le x \le 3$.

Remark : In fact when $f(x) \ge 0$, $\mathrm{A}(x) = \int_a^x f(t)dt$ represents the area bounded by $y = f(x)$, the x – axis and the ordinates $x = a$ and $x = b$. See Figure 4.1.

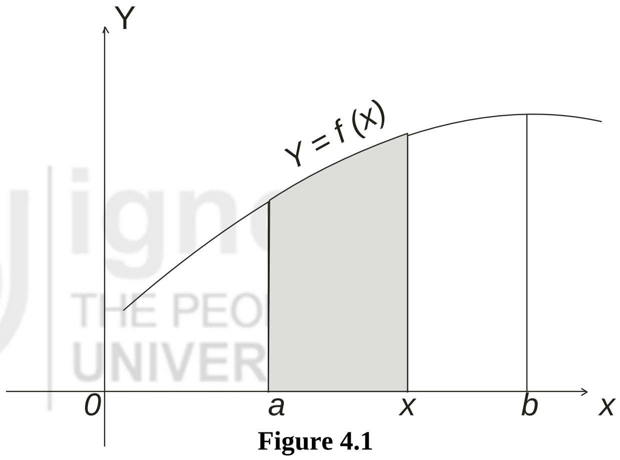


Figure 4.1

We illustrate this fact in the following illustration.

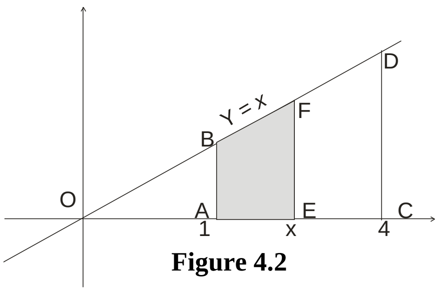


Figure 4.2

Illustration

Let $f(x) = x$, $1 \le x \le 4$.

Area of the trapezium ABFE $= \frac{1}{2}$(sum of parallel sides) × (height)

$= \frac{1}{2}$(AB + EF) (AE)

$$= \frac{1}{2}(1 + x)(x - 1) = \frac{1}{2}(x^2 - 1)$$

Also, $\mathrm{A}(x) =$

$$\int_1^x t dt = \frac{1}{2}\left[t^2\right]_1^x = \frac{1}{2}(x^2 - 1)$$

Thus, $\mathrm{A}(x)$ = area bounded by $y = x$, the *x-axis* and the ordinates $x = 1$, $x = 4$.

Solved Examples

Example 1 : Evaluate the definite integral

$$\int_1^4 \frac{dx}{\sqrt{x}}$$

Solution : We have

$$\int_1^4 \frac{dx}{\sqrt{x}} = \int_1^4 x^{-1/2}\,dx = \left[\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}\right]_1^4$$

$$= \left[2\sqrt{x}\right]_1^4 = 2(\sqrt{4} - \sqrt{1})$$

$$= 2(2-1) = 2$$

Example 2 : Evaluate the definite integral

$$\int_3^5 \frac{x}{x^2-5}\,dx$$

Solution : We first evaluate the integral

$$\int \frac{x}{x^2-5}\,dx$$

We put $x^2 - 5 = t$, so that $2x\,dx = dt$.

Thus,

$$\int \frac{x\,dx}{x^2-5} = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\log|t| = \frac{1}{2}\log|x^2-5|$$

$$\Rightarrow \int_3^5 \frac{x\,dx}{x^2-5} = \left[\frac{1}{2}\log|x^2-5|\right]_3^5$$

$$= \frac{1}{2}[\log(25-5) - \log(9-5)]$$

$$= \frac{1}{2}\log\frac{20}{4} = \frac{1}{2}\log 5$$

Example 3 Evaluate the definite integral

$$\int_1^3 \frac{dx}{x^2(x+1)}$$

Solution

We first evaluate the integral

$$\int \frac{dx}{x^2(x+1)}$$

Towords this end, we first split $\frac{1}{x^2(x+1)}$ into partial fractions

Write $$\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$$

$$\Rightarrow\ 1 = Ax(x+1) + B(x+1) + Cx^2$$

Put $x = 0$ and $x = -1$, to obtain

1 = B and 1 = C.

Next, let us compare coefficient of x^2 to obtain

$0 = A + C \Rightarrow A = -C = -1$

Thus,

$$\int \frac{dx}{x^2(x+1)} = \int \left[-\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x+1}\right] dx$$

$$= -\log|x| - \frac{1}{x} + \log|x+1|$$

$$= \log\left|\frac{x+1}{x}\right| - \frac{1}{x}$$

$$\therefore \int_1^3 \frac{dx}{x^2(x+1)} = \left[\log\left|\frac{x+1}{x}\right| - \frac{1}{x}\right]_1^3$$

$$= \log\frac{4}{3} - \frac{1}{3} - \log 2 + 1$$

$$= \log\left(\frac{2}{3}\right) + \frac{2}{3}$$

Example 4 : Evaluate the definite integral

$$\int_{-3}^{-1} \frac{dx}{x}$$

Solution : We have

$$\int_{-3}^{-1} \frac{dx}{x} = [\log|x|]_{-3}^{-1}$$

$$= \log 1 - \log 3 = \log\left(\frac{1}{3}\right)$$

Example 5 : Evaluate the definite integral

$$\int_{2}^{4} \frac{dx}{x^2 - 9}$$

Solution : Recall

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log \left[\frac{x+a}{x-a}\right]$$

Thus,

$$\int_{2}^{4} \frac{dx}{x^2 - 9} = \frac{1}{(2)(3)} \left[log \left|\frac{x+3}{x-3}\right|\right]_{2}^{4}$$

$$= \frac{1}{6} [\log 7 - log 5]$$

$$= \frac{1}{6} \log \frac{7}{5}$$

Example 6 : Evaluate the definite integral

$$\text{I} = \int_{-1}^{2} \frac{x}{(x^2 + 1)^2}\, dx$$

Solution : We first evaluate the integral

$$\text{I} = \int \frac{x}{(x^2 + 1)^2}\, dx$$

Put $x^2 + 1 = t \Rightarrow 2xdx = dt$

$$\therefore \text{ I} = \frac{1}{2} \int \frac{\text{dt}}{t^2} = \frac{1}{2} \int \text{t}^{-2}\, \text{dt}$$

$$= \frac{1}{2} \frac{\text{t}^{-2+1}}{-2+1} = -\frac{1}{2\text{t}} = -\frac{1}{2(x^2 + 1)}$$

Thus,

$$\int_{-1}^{2} \frac{x}{(x^2 + 1)^2}\, dx = \left[-\frac{1}{2(x^2 + 1)}\right]_{-1}^{2}$$

$$= -\frac{1}{2(4+1)} + \frac{1}{2(1+1)}$$

$$= \frac{1}{4} - \frac{1}{10} = \frac{3}{20}$$

Example 7: Evaluate the definite integral

$$\int_1^5 \frac{x}{\sqrt{2x-1}}\, dx$$

Solution : We first evaluate the integral

$$\mathrm{I} = \int \frac{x}{\sqrt{2x-1}}\, dx$$

We put $2x - 1 = t^2 \Rightarrow x = \frac{1}{2}(t^2+1)$

$$\Rightarrow dx = \tfrac{1}{2}(2t)dt = tdt$$

Thus,

$$\mathrm{I} = \int \frac{\frac{1}{2}(t^2+1)}{t}\, t\, \mathrm{dt} = \frac{1}{2}\int (t^2+1)\, dt$$

$$= \frac{1}{2}\left[\frac{1}{3}t^3 + t\right] = \frac{1}{2}\left[\frac{1}{3}\,(2x-1)^{3/2} + \sqrt{2x-1}\right]$$

$$\therefore \int_1^5 \frac{x}{\sqrt{2x-1}}\, dx = \left[\frac{1}{6}(2x-1)^{\frac{3}{2}} + \frac{1}{2}(2x-1)^{\frac{1}{2}}\right]_1^5$$

$$= \frac{1}{6}\left[9^{\frac{3}{2}} - 1\right] + \frac{1}{2}\left[9^{\frac{1}{2}} - 1\right]$$

$$= \frac{1}{6}(26) + \frac{1}{2}(2)$$

$$= \frac{13}{3} + 1 = \frac{16}{3}$$

Example 8 : Evaluate the definite integral

$$\int_0^1 \frac{24x^3}{(1+x^2)^4}\, dx$$

Solution : We first evaluate the integral

$$\mathrm{I} = \int \frac{24x^3}{(1+x^2)^4}\, dx = 12\int \frac{2x^2}{(1+x^2)^4}\, x\, dx$$

Put $1 + x^2 = t \Rightarrow 2xdx = \mathrm{dt}$,

So, that

$$\mathrm{I} = 12\int \frac{(t-1)}{t^4}\, dt = 12\int (t^{-3} - t^{-4})\, dt$$

$$= 12\left[\frac{t^{-2}}{-2} - \frac{t^{-3}}{-3}\right] = 12\left(\frac{1}{3t^3} - \frac{1}{2t^2}\right)$$

$$= \frac{4}{t^3} - \frac{6}{t^2} = \frac{4}{(1+x^2)^3} - \frac{6}{(1+x^2)^2}$$

$$\text{Thus, } \int_0^1 \frac{24x^3}{(1+x^2)^4}\, dx = \left[\frac{4}{(1+x^2)^3} - \frac{6}{(1+x^2)^2}\right]_0^1$$

$$= \frac{4}{2^3} - \frac{6}{2^2} - 4 + 6$$

$$= \frac{1}{2} - \frac{3}{2} + 2 = 1$$

Check Your Progress – 1

Evaluate the following definite integrals.

1. $\displaystyle\int_0^2 \frac{x}{x+1}\, dx$

2. $\displaystyle\int_1^2 \frac{dx}{(x+1)(x+2)}$

3. $\displaystyle\int_0^3 (e^x + x)\quad dx$

4. $\displaystyle\int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$

5. $\displaystyle\int_0^2 x\sqrt{3x-2}\, dx$

6. $\displaystyle\int_0^2 \frac{dx}{x+4-x^2}$

7. $\displaystyle\int_a^b \frac{logx}{x}\, dx$

8. $\displaystyle\int_0^a \frac{x^3}{\sqrt{a^2-x^2}}\, dx$

Answers

1. 2 – log2

2. $\log\left(\frac{9}{8}\right)$

3. $(e^3 - 1) + 9/2$

4. 1

5. 326/135

6. $\dfrac{1}{\sqrt{17}} \log\left(\dfrac{21-5\sqrt{17}}{4}\right)$

7. $\dfrac{1}{2}\left(\log\dfrac{b}{a}\right) \log ab$

8. $\dfrac{2}{3}a^3$

4.3 AREA UNDER THE CURVE

Suppose f is continuous and

$f(x) \geq 0\ \forall\ x \in [a,b]$, then area bounded by $y = f(x)$, the x – *axis* and the ordinates $x = a$, and $x = b$ is given by

$$\int_a^b f(x)dx$$

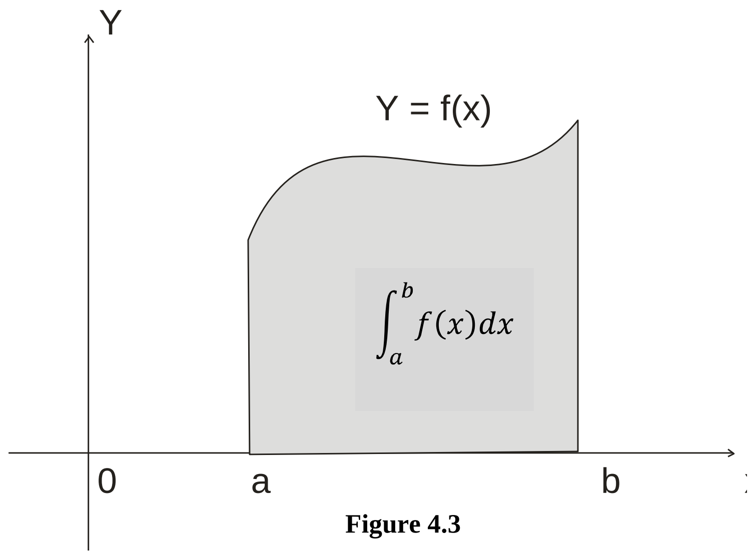


Figure 4.3

Area between two Curves

Suppose f and g be two continuous functions defined on $[a,b]$ such that $f(x) \geq g(x)\ \forall\ x \in [a,b]$ then area bounded by $y = f(x)$, $y = g(x)$ and the ordinates $x = a$ and $x = b$, is

$$\int_a^b [f(x) - g(x)]dx$$

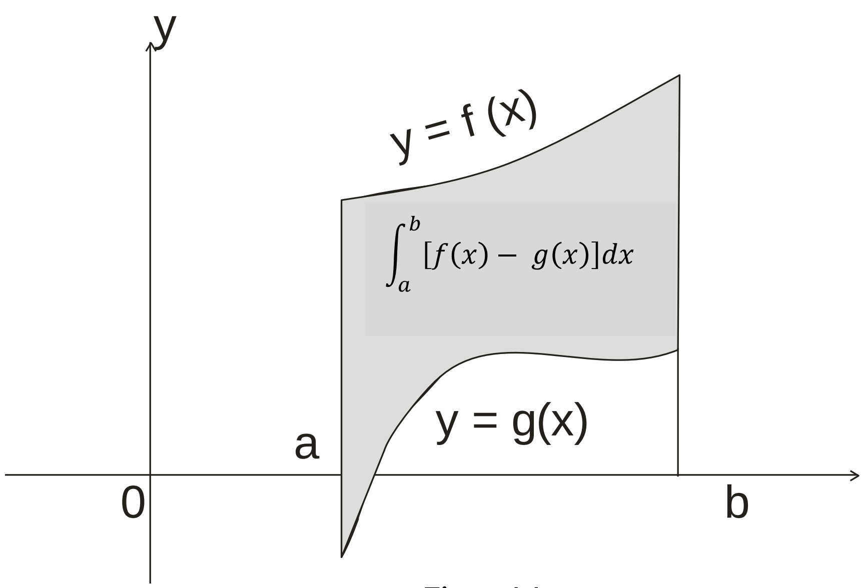


Figure 4.4

Solved Examples

Example 9 : Find the area bounded by the x –axis and $y = 2x - 3$, $2 \leq x \leq 5$.

Solution : $y = 2x - 3$ represents a straight line *passing* through (2, 1) and (5, 7)

Required area

$$= \int_2^5 (2x - 3)dx$$

$$= [x^2 - 3x]_2^5$$

$$= 25 - 3(5) - (4 - 6)$$

= 12 sq. units

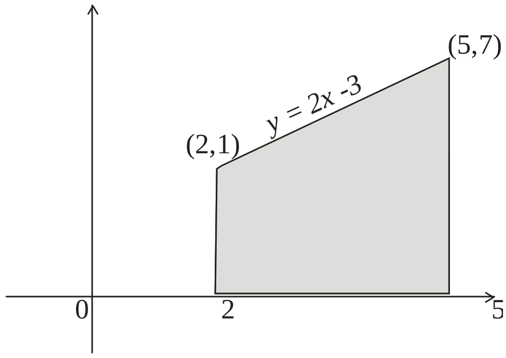


Figure 4.5

Example 10: Find the area bounded by the x–*axis*, $y = x^2$, $0 \leq x \leq 3$.

Solution $y = x^2$ represents a parabola that open upwards.
Required area

$$= \int_0^3 x^2 \, dx$$

$$= \left[\frac{1}{3}x^3\right]_0^3 = 9 \text{ sq. units}$$

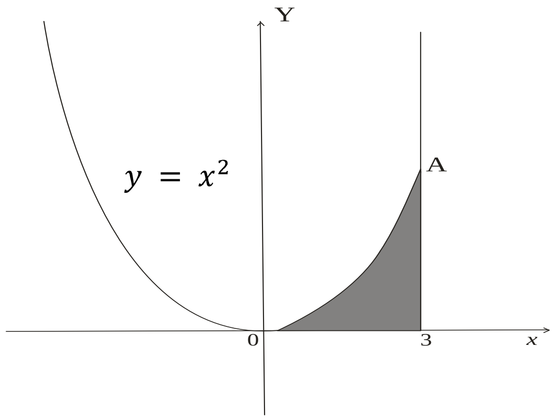


Figure 4.6

Example 11: Find the area bounded by the x– *axis*, $y = \sqrt{x}$ and $1 \leq x \leq 4$.

Solution : $y = \sqrt{x}$ represents part of parabola whose axis is the x-axis.

Required area

$$= \int_1^4 \sqrt{x}dx = \int_1^4 x^{1/2}dx$$

$$= \left[\frac{x^{3/2}}{3/2}\right]_1^4 = 2/3\,(4^{3/2} - 1^{3/2})$$

$$= \frac{14}{3} \text{sq. units}$$

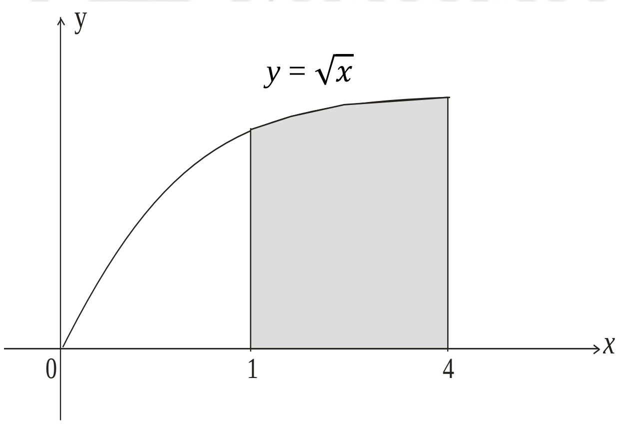


Figure 4.7

Example 12 Find the area bounded by the curves y = x^2 and $y = x$.

Solution : To obtain point of intersection of y = x^2 and $y = x$, we set

$x^2 = x \Rightarrow x(x-1) = 0 \Rightarrow x = 0, 1.$

Thus, the two curves intersect in (0,0) and (1,1). See Fig 4.8

Required area

$= \int_0^1 (x - x^2)dx$

$= \left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^1$

$= \left(\frac{1}{2} - \frac{1}{3}\right)$ sq. units

$= \frac{1}{6}$ sq. units.

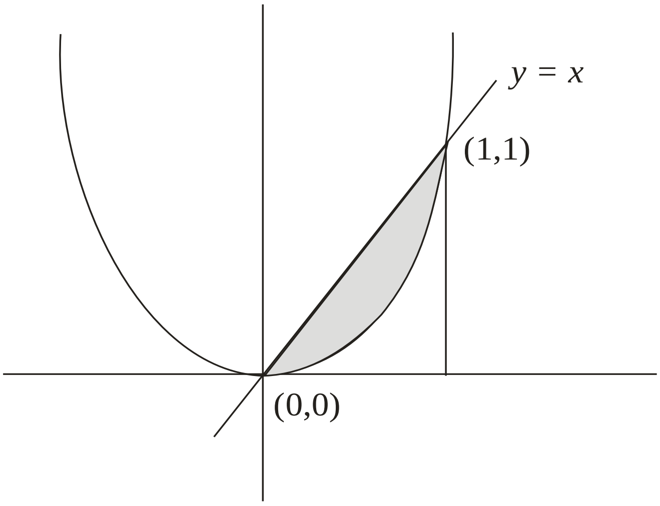


Figure 4.8

Example 13 : Find the area bounded by the curves $y = \sqrt{x}$ *and* $y = x$.

Solution : To obtain the point of intersection,

We set $\sqrt{x} = x \Rightarrow \sqrt{x}(1 - \sqrt{x}) = 0$

$\Rightarrow \sqrt{x} = 0$ or $1 - \sqrt{x} = 0 \Rightarrow x = 0$ or $x = 1$.

Required area

$= \int_0^1 (\sqrt{x} - x)dx$

$= \left[\frac{x^{3/2}}{3/2} - \frac{x^2}{2}\right]_0^1$

$= \left(\frac{2}{3} - \frac{1}{2}\right)$sq.units

$= \left(\frac{1}{6}\right)$sq.units

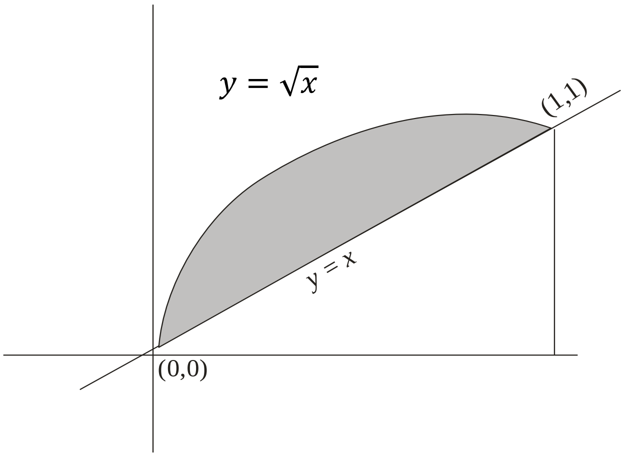


Figure 4.9

Example 14 : Find the area lying between

$y = 2x + 1, y = 3x, \quad 1 \leq x \leq 4.$

Solution : The two straight lines intersect in (1,3)

Required area

$= \int_1^4 [3x - (2x+1)]\, dx$

$= \int_1^4 [(x-1)dx = \left[\frac{1}{2}x^2 - x\right]_1^4 \, dx$

$= \left[\frac{1}{2}(4^2) - 4 - \frac{1}{2}(1^2) + 1\right]$ sq. units.

= 4.5 sq. units.

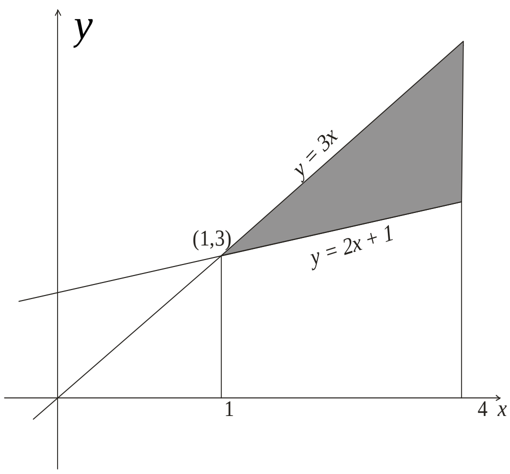


Figure 4.10

Example 15: Find the area bounded by the x – axis, $y = \frac{1}{x}$, $1 \le x \le 4$.

Solution :

Required area $= \int_1^4 \frac{1}{x}\, dx$

$= \log x \Big|_1^4$

= log 4 – log 1

= 2log 2 sq. units

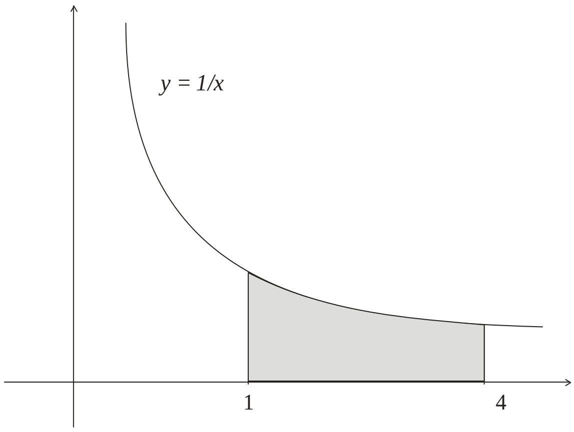


Figure 4.11

Check Your Progress – 2

1. Find the area bounded by the x–axis, $y = 5 - 2x$ and the y – axis.

2. Find the area bounded by the x–axis, $y = 2 + 3x$ and the ordinates $x = 0$ and $x = 3$

3. Find the area bounded lying between the lines $y = 3 + 2x$, $y = 3 - x$, $0 \le x \le 3$.

4. Find the area bounded by the curves y = x^2 and $y^2 = x$.

5. Find the area bounded by the line $y = x$ and the parabola $y^2 = x$.

6. Find the area bounded by the curve y = e^x, the x–axis and the ordinates $x = 1$ and $x = 3$.

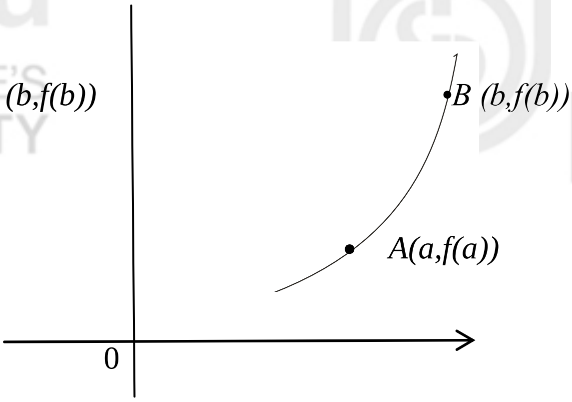
7. Find the area lying in the first quadrant and bounded by the x-axis, the line $y = x$ the curve $y = 1/x$ and the ordinate $x = 2$.

4.4 LENGTH OF CURVES (RECTIFICATION)

To find length of curve

Y = *f(x)*, from the point of A*(a, f(a))* and B *(b,f(b))*

$$= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\; dx$$

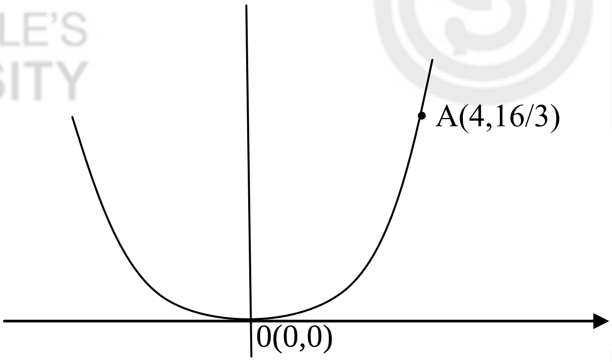


Solved Examples

Example 16 Find the length of the curve

$$y = \frac{2}{3}x^{3/2} \text{ from } (0,0) \text{to } (4, 16/3)$$

Solution : We have



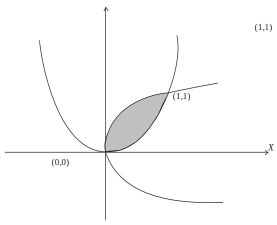
$$\frac{dy}{dx} = \frac{2}{3}.\frac{3}{2}x^{1/2} = x^{1/2}$$

Required length of the curve

$$= \int_0^4 \sqrt{1 + (x^{1/2})^2}\; dx$$

$$\int_0^4 \sqrt{1+x}dx = \left[\frac{2}{3}(1+x)^{3/2}\right]_0^4$$

$$= \frac{2}{3}\left[5\sqrt{5} - 1\right] \; units$$



Example 17 : Find the length of the curve y = 2*x* + 3

Solution : We have

$$\frac{dy}{dx} = 2$$

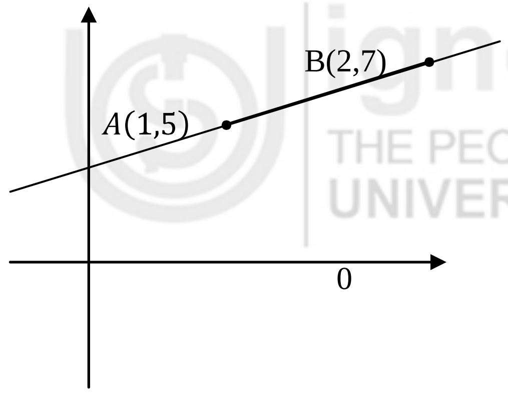
Required length

$$= \int_1^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_1^2 \sqrt{1+4}\ dx$$

$$= \left[\sqrt{5}\ x\right]_1^2$$

$$= 2\sqrt{5} - \sqrt{5} = \sqrt{5}\ units$$



Check Your Progress – 3

1. Find the length of the curve y = 3 – x from (–1, 4) to (3,0).
2. Find the length of the curve y = 3 + $\frac{1}{2}x$ from (0,3) to (2, 4).
3. Find the length of the curve y = $2x^{3/2}$ from point (1,2) to (4,16).

4.7 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress 1

1. $\int_0^2 \frac{x}{x+1}\,dx = \int_0^2 \frac{x+1-1}{x+1}\,dx \int_0^2 \left[1 - \frac{1}{x+1}\right]dx$

$$= \left. x - \log(x+1)\right|_0^2$$

$$= [2 - \log 2] - [0 - \log 1]$$

$$= 2 - \log 2$$

2. $\int_1^2 \frac{dx}{(x+1)(x+2)} = \int_0^2 \left[\frac{1}{x+1} - \frac{1}{x+2}\right] dx$

[split into partial fractions]

$$= \left[\log\left|\frac{x+1}{x+2}\right|\right]_1^2 = \log\frac{3}{4} - \log\frac{2}{3}$$

$$= \log\left(\frac{9}{8}\right)$$

3. $\int_0^3 (e^x + x)dx = \left[\left(e^x + \frac{1}{2}x^2\right)\right]_0^3$

$$= (e^3 - 1) + \frac{9}{2}$$

4. To evaluate $\quad \mathrm{I} = \int \frac{x}{\sqrt{1-x^2}}\,dx$

put $1 - x^2 = t^2$, so that $-x dx = t\,dt$, and

$$I = \int \frac{-t}{t}\,dt = -\int dt = -t = -\sqrt{1-x^2}$$

$$Thus, \int_0^1 \frac{x}{\sqrt{1-x^2}}\,dx = \left[-\sqrt{1-x^2}\,\right]_0^1 = -0 + 1 = 1$$

5. To evaluate $\quad \mathrm{I} = \int x\sqrt{3x-2}\,dx$, put

$3x - 2 = t^2$, so that $3dx = 2t\,\mathrm{d}t$

$$\therefore I = \int \frac{1}{3}\,(t^2 + 2)\frac{2}{3}t.tdt = \frac{2}{9}\int (t^4 + 2t^2)dt$$

$$= \frac{2}{9}\left(\frac{1}{5}t^5 + \frac{2}{3}\,t^3\right)$$

$$= \frac{2}{45}\,(3x-2)^{\frac{5}{2}} + \frac{4}{27}(3x-2)^{\frac{3}{2}}$$

Thus, $\int_1^2 x\sqrt{3x-2}\,\,dx$

$$= \left[\frac{2}{45}(3x-2)^{\frac{5}{2}} + \frac{4}{27}(3x-2)^{\frac{3}{2}}\right]_1^2$$

$$= \frac{2}{45}\left(4^{5/2} - 1^{5/2}\right) \quad + \frac{4}{27}\left(4^{3/2} - 1^{3/2}\right)$$

$$= \frac{2}{45}(2^5 - 1) \quad + \frac{4}{27}(8-1)$$

$$= \frac{62}{45} + \frac{28}{27} = \frac{326}{135}$$

6. We have $x + 4 - x^2 = -(x^2 - x - 4)$

$$= -\left[x^2 - 2\left(\frac{1}{2}\right) + \frac{1}{4} - 4 - \frac{1}{4}\right]$$

$$= -\left[\left(x-\frac{1}{2}\right)^2 - \frac{17}{4}\right] = \frac{17}{4} - \left(x-\frac{1}{2}\right)^2\Bigg]$$

We have

$$\int_0^2 \frac{dx}{x+4-x^2} = \int_0^2 \frac{dx}{\frac{17}{4}-(x-\frac{1}{2})^2}$$

$$= \left[\frac{1}{\sqrt{17}} log \left|\frac{\sqrt{\frac{17}{4}}-(x-\frac{1}{2})^2}{\sqrt{\frac{17}{4}}+(x-\frac{1}{2})^2}\right|\right]_0^2$$

$$= \frac{1}{\sqrt{17}} \log \left|\frac{(\sqrt{17}-3)/2}{(\sqrt{17}+3)/2}\right| - \log \left|\frac{(\sqrt{17}+1)/2}{(\sqrt{17}-1)/2}\right|$$

$$= \frac{1}{\sqrt{17}} \log \left|\frac{(\sqrt{17}-3)(\sqrt{17}-1)}{(\sqrt{17}+3)(\sqrt{17}+1)}\right|$$

$$= \frac{1}{\sqrt{17}} \log \left|\frac{17+3-4\sqrt{17}}{17+3+4\sqrt{17}}\right| = \log \left|\frac{5-\sqrt{17}}{5+\sqrt{17}}\right|$$

$$= \frac{1}{\sqrt{17}} \log \left[\frac{(5-\sqrt{17})^2}{25-17}\right]$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{25+17-10\sqrt{17}}{8}\right)$$

$$= \frac{1}{\sqrt{17}} \log \left(\frac{21-5\sqrt{17}}{4}\right)$$

7. Put $\log x = t$, so that $\frac{1}{x}\,dx = dt$.

$$\therefore \int \frac{\log x}{x}\,dx = \int t\,dt = \frac{1}{2}\,t^2 = \frac{1}{2}\,(logx)^2$$

Thus,

$$\int_a^b \frac{\log x}{x}\,dx = \left[(\log x)^2\right]_b^a$$

$$= \frac{1}{2}[(\log b)^2 - (\log a)^2]$$

$$= \frac{1}{2}\,[\log b - \log a)\,(\log b + \log a)]]$$

$$= \frac{1}{2}(\log \frac{b}{a})\,(\log(ab)).$$

8. Put $a^2 - x^2 = t^2$, so that $-2x\,dx = 2t\,dt$

$$\int \frac{x^3}{\sqrt{a^2 - x^2}}\,dx = \int \frac{x^2}{\sqrt{a^2 - x^2}}\,x\,dx$$

$$= \int \frac{a^2 - t^2}{t}\,(-t)\,dt$$

$$= \int (t^2 - a^2)\,dt = \frac{1}{3}t^3 - a^2 t$$

$$= \frac{1}{3}(a^2 - x^2)^{3/2} - a^2(a^2 - x^2)^{1/2}$$

Thus,

$$\int_0^a \frac{x^3}{\sqrt{a^2 - x^2}}\,dx$$

$$= \left[\frac{1}{3}(a^2 - x^2)^{3/2} - a^2(a^2 - x^2)^{1/2}\right]_0^a$$

$$= 0 - \frac{1}{3}a^3 + a^3 = \frac{2}{3}a^3$$

Check Your Progress – 2

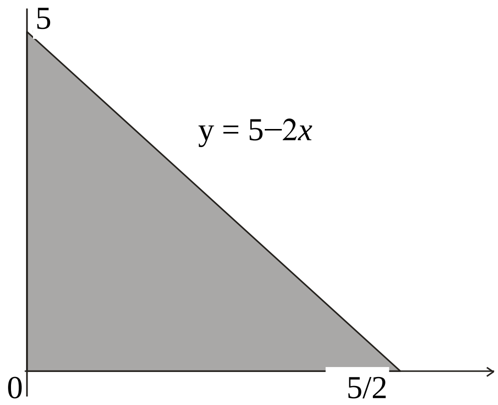
1. The line $y = 5 - 2x$ meets the x–axis at (5/2, 0) and the y–axis at (0,5).

Required area

$$= \int_0^{5/2} (5 - 2x)\,dx$$

$$= \left[5x - x^2\right]_0^{5/2}$$

$$= \frac{25}{4} \text{ sq. units}$$

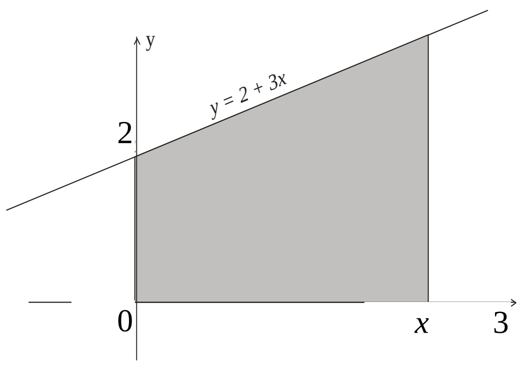


2. Required area

$$= \int_0^3 (2 + 3x)\,dx$$

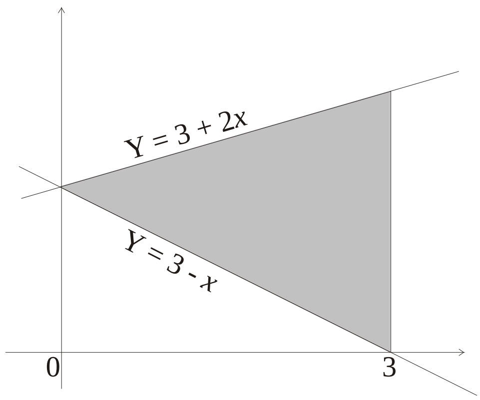
$$= \left[2x + \frac{3}{2}x^2\right]_0^3$$

$$= 6 + \frac{3}{2}(9) = \frac{39}{2} \text{ sq. units}$$



3. $y = 3 + 2x$ represents a straight line passing through (0,3) and sloping upwards, and $y = 3 - 2x$ represents a straight line passing through (0,3) and sloping downwards and meeting the x –axis at (3,0)

Required area



$$= \int_0^3 [3 + 2x - (3 - x)]dx$$

$$= \int_0^3 3x dx = \left[\frac{3}{2}x^2\right]_0^3 = \frac{27}{2} \text{ sq. units}$$

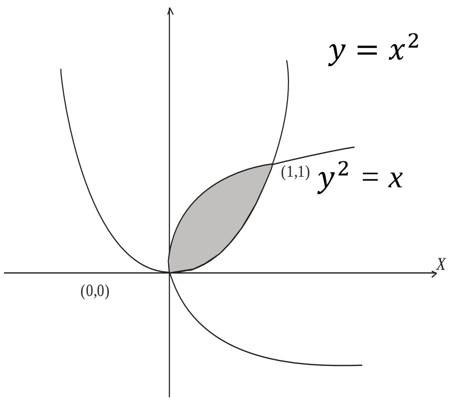
4. We first find the points of intersection of $y = x^2$ and $y^2 = x$. We have $x = y^2 = (x^2)^2$

$\Rightarrow x = x^4$

$\Rightarrow x(1 - x^3) = 0$

$\Rightarrow x = 0$ or $x = 1$

Required area



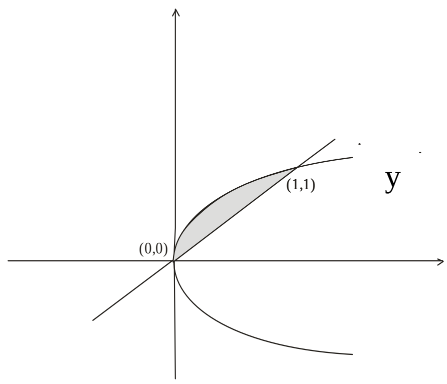
$$= \int_0^1 (\sqrt{x} - x^2)\, dx = \left[\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3} \text{ sq. units}$$

5. We first find the point of intersections of $y = x$ and $y^2 = x$. We have $x^2 = x$

$\Rightarrow x(x - 1) = 0$

$\Rightarrow x = 0$ or $x = 1$

Required area



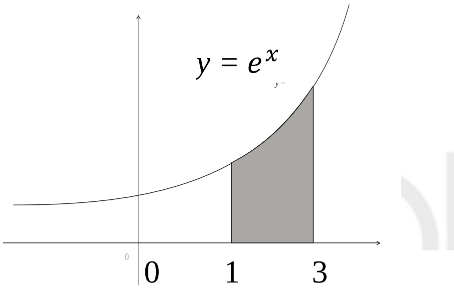
$$= \int_0^1 (\sqrt{x} - x)\, dx = \left[\frac{x^{3/2}}{\frac{3}{2}} - \frac{x^3}{3} \right]_0^1$$

$$= \frac{2}{3} - \frac{1}{3} = \frac{1}{6} \text{ sq. units}$$

6. Required area

$$= \int_1^3 e^x\, dx$$

$$= \left[e^x \right]_1^3 = (e^3 - e)\ sq.units$$



7. To obtain the points of intersection of

$y = x$ and $y = 1/x$,

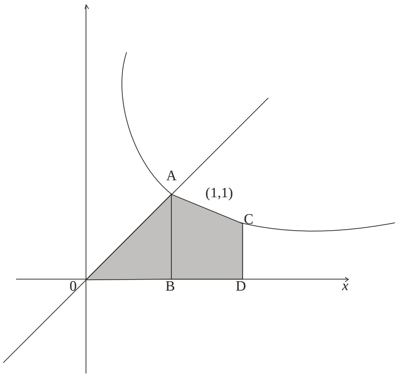
We set

$x = 1/x$

$\Rightarrow x^2 = 1$

$\Rightarrow x = \pm 1$

We take $x = 1$, as we are interested in the a



Required area

= area (OAB) + area (BACD)

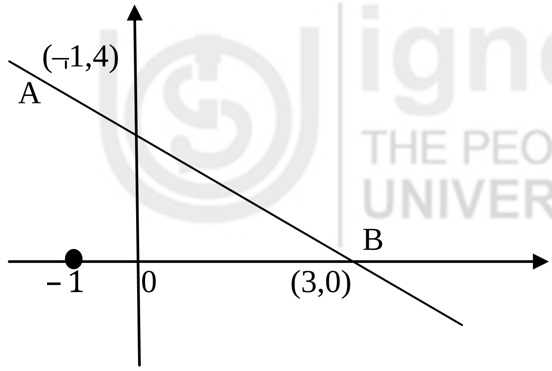
$$\int_0^1 x\ dx + \int_1^2 \frac{1}{x}\ dx$$

$$= \left[\frac{1}{2} x^2 \right]_0^1 + \log x\ \Big|_1^2$$

$$= \left(\frac{1}{2} + \log 2 \right) \text{sq.units}$$

Check Your Progress – 3

1. We have



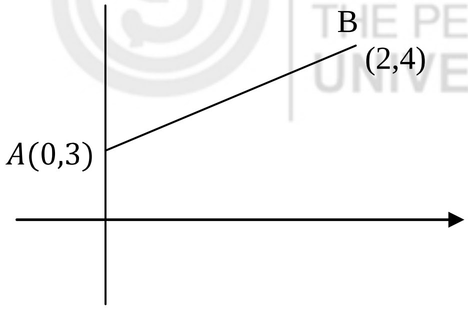
$$\frac{dy}{dx} = -1$$

Required length

$$= \int_{-1}^{3} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_{1}^{3} \sqrt{1+\ 1}\ dx = \sqrt{2}\ x\Big|_{-1}^{3} = \sqrt{2}\,(3+1) = 4\sqrt{2} \text{ units}$$

2. We have



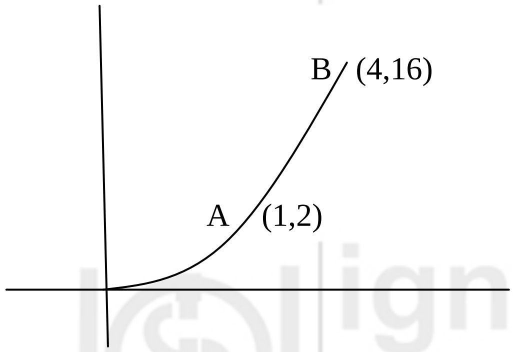
$$\frac{dy}{dx} = \frac{1}{2}$$

Required length

$$= \int_{0}^{2} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

$$= \int_{0}^{2} \sqrt{1 + \frac{1}{4}}\ dx = \frac{\sqrt{5}}{2}\int_{0}^{2} dx = \frac{\sqrt{5}}{2}\ x\Big|_{0}^{2} = \sqrt{5} \text{ units}$$

3. We have



$$\frac{dy}{dx} = 3x^{1/2}$$

Required length

$$= \int_{1}^{4} \sqrt{1+\ 9x}\ dx = \left[\frac{(1+9x)^{3/2}}{9\left(\frac{3}{2}\right)}\right]_{1}^{4} = \frac{2}{27}\left[37\sqrt{37} - 10\sqrt{10}\right] \text{units}$$

4.6 SUMMARY

In this unit, as the title of the unit suggests, applications of Integral Calculus, are discussed. In **section 4.2**, first, the concept of 'definite integral' is introduced and then methods of finding the value of a definite integral, are illustrated through examples. Methods of finding the area under a curve is illustrated in **section 4.3**. Method of finding the length of a curve is discussed in **section 4.4** .

Answers/Solutions to questions/problems/exercises given sections of the unit are available in **section 4.5**.