UNIT 2 MATRICES - I

Structure

2.0 INTRODUCTION

In this Unit, we shall learn about Matrices. Matrices play central role in mathematics in general, and algebra in particular. A matrix is a rectangular array of numbers. There are many situations in mathematics and science which deal with rectangular arrays of numbers. For example, the following table gives vitamin contents of three food items in conveniently chosen units.

	Vitamin A	Vitamin C	Vitamin D
Food I	0.4	0.5	0.1
Food II	0.3	0.2	0.5
Food III	0.2	0.5	0

The above information can be expressed as a rectangular array having three rows and three columns.

$$\begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.3 & 0.2 & 0.5 \\ 0.2 & 0.5 & 0 \end{bmatrix}$$

The above arrangement of numbers is a matrix of order 3×3. Matrices have become an important an powerful tool in mathematics and have found applications to a very large number of disciplines such as Economics, Physics, Chemistry and Engineering.

In this Unit, we shall see how Matrices can be combined thought the arithematic operations of addition, subtraction, and multiplication. The use of Matrices in solving a system of linear equations will also be studied. In Unit 1 we have already studied determinant. It must be noted that a matrix is an arrangement of numbers whereas determinant is number itself. However, we can associate a determinant to every square matrix i.e., to a matrix in which number of rows is equal to the number of columns.

2.1 OBJECTIVES

After studying this Unit, you should be able to :

- define the term matrix;
- add two or more Matrices;
- multiply a matrix by a scalar;
- multiply two Matrices;
- find the inverse of a square matrix (if it exists); and
- use the inverse of a square matrix in solving a system of linear equations.

2.2 MATRICES

We define a matrix as follows :

Def : A $m \times n$ matrix A is a rectangular array of $m\ n$ real (or complex numbers) arranged in m horizontal rows an n vertical columns :

$$A = \begin{array}{c@{\quad}l} \left(\begin{array}{cccccc} a_{11} & a_{12} & \dots\dots\dots & a_{1j} & \dots\dots\dots & a_{1n} \\ a_{21} & a_{22} & \dots\dots\dots & a_{2j} & \dots\dots\dots & a2_{n} \\ : & : & : & : & & \\ : & : & : & : & & \\ a_{i1} & a_{i2} & \dots\dots\dots & a_{ij} & \dots\dots\dots & ai_{n} \\ : & : & & & & \\ : & & & & & \\ a_{m1} & a_{m2} & \dots\dots\dots & a_{mj} & \dots\dots\dots & a_{mn} \end{array}\right) & \begin{array}{l} \\ \\ \\ \\ ith\ row \\ \\ \ldots(1) \\ \\ \end{array} \\ \begin{array}{cccccc} \phantom{a_{11}} & \phantom{a_{12}} & & \begin{array}{c} jth \\ column \end{array} & & \phantom{a_{1n}} \end{array} & \end{array}$$

As it is clear from the above definition, the ith row of A is $(a_{ij}\ \ a_{i2}\ \ \ldots\ a_{in})$ $(1 \leq i \leq m)$ and the jth column is

$$\begin{pmatrix} a_{1\mathrm{j}} \\ a_{2\mathrm{j}} \\ : \\ : \\ a_{mj} \end{pmatrix} \qquad (1 \leq j \leq n)$$

We also note that each element a_{ij} of the matrix has two indices : the row index i and the column index j. a_{ij} is called the $(i,j\ )th$ element of the matrix. For convenience, the Matrices will henceforward be denoted by capital letters and the elements (also called entries) will be denoted by the corresponding lower case letters.

The matrix in (1) is often written in one of the following forms :

A = *[aij]; * A = *(aij)*, A = $(a_{ij})_{m\times n}$ *or* A = $(a_{ij})_{m\times n}$

With i = 1, 2, ………….., m and j = 1, 2, ………., n

The **dimension** or **order** of a matrix A is determined by the number of rows and columns of the matrix. If *a* matrix A has *m* rows and *n* columns we denote its dimension or order by $m\times n$ read *"m by n"*.

For example, A = $\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}$ is a 2 × 2 matrix and B = $\begin{bmatrix}1 & 0 & 1\\2 & 1 & 3\end{bmatrix}$ is a 2 × 3 order matrix.

Note a that an $m \times n$ matrix has mn elements.

Type of Matrices

1. **Square Matrix :** A **square matrix** is one in which the number of rows is equal to the number of columns. For instance,

   A = $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$, B = $\begin{bmatrix}3 & -1 & 6\\8 & 2 & 9\\0 & 7 & 3\end{bmatrix}$, C = $\begin{bmatrix}5 & -1 & 8 & 3\\2 & 5 & 7 & 8\\3 & 6 & 11 & 0\\0 & -1 & 8 & 7\end{bmatrix}$

   are square Matrices.

   If a square matrix has *n* rows (and thus *n* columns), then *A* is said to be a square matrix of order *n*.

2. **Diagonal Matrix :** A square matrix A$[a_{ij}]_{n\times n}$ for which $a_{ij} = 0$ for $i \neq j,$ is called a **diagonal matrix**.

   For instance,

   D= $\begin{bmatrix}4 & 0 & 0\\0 & -2 & 0\\0 & 0 & 6\end{bmatrix}$ and E = $\begin{bmatrix}10 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 3 & 0\\0 & 0 & 0 & 5\end{bmatrix}$

   are diagonal Matrices.

   If A = $[a_{ij}]_{n\times n}$ is a square matrix of order *n*, then the numbers a_{11}, a_{22}, … a_{nn} are called diagonal elements, and are said to form the **main diagonal** of A. Thus, a square matrix for which every term off the main diagonal is zero is called a diagonal matrix.

3. **Scalar Matrix :** A diagonal matrix A = $[a_{ij}]_{n\times n}$ for which all the terms on the main diagonal are equal, that is $a_{ij} = k$ for $i = j$ and $a_{ij} = 0$ for $i \neq j$ is called a **scalar matrix.**

For instance

$$H = \begin{bmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & -5 \end{bmatrix} \text{ and } I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

are scalar Matrices.

4. **Unit or Identity Matrix :** A square matrix $A = [a_{ij}]_{n\times n}$ is said to be the **unit matrix** or **identity matrix** if

$$a_{ij} = \begin{cases} 0 & if\ i \neq j \\ 1 & if\ i = j \end{cases}$$

Note that a unit matrix is a scalar matrix with is on the main diagonal.
We denote the unit matrix having n rows (and n columns) by In.
For example,

$$I_3 = [1],\ I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

5. **Row Matrix or Column Matrix :** A matrix with just one row of elements is called a **row matrix or row vector**. While a matrix with just one column of elements is called a **column matrix** or **column vector.**

For instance, A = $[2 \ \ 5 \ -15]$ is a row matrix whereas $B = \begin{bmatrix} 3 \\ 5 \\ 7 \end{bmatrix}$

is a column matrix.

6. **Zero matrix or Null matrix :** An $m \times n$ matrix is called a **zero matrix** or **null matrix** if each of its elements is zero.

We usually denote the zero matrix by $O_{m \times n}$

$O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ are example of zero matrices.

Equality of Matrices

Let A = $[a_{ij}]_{m\times n}$ and B =$[b_{ij}]_{r\times s}$ be two Matrices. We say that A and B are equals if

1. $m = r$, i.e., the number of rows in A equals the number of rows in B.

2. $n = s$, *i.e.*, the number of columns in A equals the number of columns in B.
3. $a_{ij} = b_{ij}$ for $I = 1, 2, \ldots\ldots m$ and $j = 1, 2, \ldots\ldots, n$.

We then write A = B, read as "matrix A is equal to B" In other words, two Matrices are equal if their order are equal and their corresponding elements are equal.

Example 1 : Let A and B be two Matrices given by $A = \begin{bmatrix} x+2 \\ 3y-7 \end{bmatrix}$,

$B = \begin{bmatrix} 4-y \\ x-3 \end{bmatrix}$. Find x and y so that $A = B$.

Solution: Both the Matrices are of order 2 ×1. Therefore, by the definition of equality of two Matrices, we have $x + 2 = 4 - y$ and $3y - 7 = x - 3$. That is, $x + y = 2$ and $x - 3y = -4$. Solving these two equations. We get $x = 1/2$ and $y = 3/2$. We can check this solution by substitution in A and B.

$$A = \begin{bmatrix} \left(\frac{1}{2}\right) + 2 \\ 3\left(\frac{3}{2}\right) - 7 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -5/2 \end{bmatrix} \quad \text{and}$$

$$B = \begin{bmatrix} 4 - \left(\frac{3}{2}\right) \\ \left(\frac{1}{2}\right) - 3 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -5/2 \end{bmatrix}$$

Transpose of a Matrix

Definition: Let $A = [a_{ij}]_{m \times n}$, be a matrix. The **transpose** of A, denoted by A´, is the matrix $A' = [a_{ij}]_{m \times n}$, where $b_{ij} = a_{ji}$ for each i and j.

The transpose of a matrix A is by definition, that matrix which is obtained from A by interchanging its rows and columns.

So, if $A = \begin{bmatrix} -3 & 2 & 5 \\ 0 & 7 & 8 \end{bmatrix}$, then

its transpose is the matrix

$$A' = \begin{bmatrix} -3 & 0 \\ 2 & 7 \\ 5 & 8 \end{bmatrix}.$$

Symmetric and Skew Symmetric Matrices

Definition : A square matrix $A = [a_{ij}]_{n \times n}$, is said to be **symmetric** if A'= A*;* it is **skew symmetric** if A' = −A.

For example A = $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ is symmetric and B = $\begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix}$ is a skew – symmetric matrix.

Check Your Progress 1

1. Construct a 2×2 matrix A = $[a_{ij}]_{2 \times 2}$ where elements are given by
   (a) $a_{ij} = \frac{1}{2}(i + 2j)^2$ (b) $a_{ij} = \frac{1}{2}(i - j)^2$

2. Find x, y when $\begin{bmatrix} x & y \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 2x + y & x - y \\ 3 & 5 \end{bmatrix}$.

3. a, b, c and d such that

$$\begin{bmatrix} a - b & 2c + d \\ 2a - b & 2a + d \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 12 & 15 \end{bmatrix}.$$

4. Find the transpose of following Matrices and find whether the matrix is symmetric or skew symmetric.

   (a) A= $\begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix}$ (b) $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$

   (c) $A = \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}$

2.3 OPERATION ON MATRICES

Addition

Let A = $[a_{ij}]_{m\times n}$ and B= $[b_{ij}]_{r\times s}$ be two Matrices. We say that A and B are comparable for addition if $m = r$ and $n = s$. That is, A and B are comparable for addition if they have same order.

We define addition of Matrices as follows :

Definition : Let A =$[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{m\times n}$ be two Matrices. The sum of A and B is the $m \times n$ matrix C =$[c_{ij}]$ such that

$$C_{ij} = a_{ij} + b_{ij} \qquad (1 \le i \le m, 1 \le j \le n).$$

That is, C is obtained by adding the corresponding elements of A and B. We usually denote C by A + B.

Note that

A + B = $[c_{ij}]_{m\times n}$ = $[a_{ij} + b_{ij}]_{m\times n}$

For example, if A $=\begin{bmatrix}0 & 0 & 1\\ 3 & 2 & 5\end{bmatrix}$ and B $=\begin{bmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{bmatrix}$, then

$$A + B = \begin{bmatrix}0 & 0 & 1\\ 3 & 2 & 5\end{bmatrix} + \begin{bmatrix}1 & 2 & 3\\ -1 & 0 & 2\end{bmatrix}$$

$$= \begin{bmatrix}0+1 & 0+2 & 1+3\\ 3-1 & 2+0 & 5+2\end{bmatrix}$$

$$= \begin{bmatrix}1 & 2 & 4\\ 2 & 2 & 7\end{bmatrix}$$

It must be noted that Matrices of different orders cannot be added. For instance,
A = $\begin{bmatrix}2 & 5\\ 3 & -1\end{bmatrix}$ and B = $\begin{bmatrix}5\\ 7\end{bmatrix}$ Cannot be added.

The following properties of matrix addition can easily be verified.

1. Matrix addition is commutative. That is, if A and B are two $m \times n$ matrices, then
$$A + B = B + A.$$
2. Matrix addition is associative. That is, if A, B and C are three $m \times n$ matrices, then

$$(A + B) + C = A + (B + C)$$

3. If A $[a_{ij}]$ = is an $m\times n$ matrix, then
$$A + O_{m\times n} = O_{m\times n} + A = A,$$
. where $O_{m\times n}$ is the $m\times n$ null matrix.

4. If A is an $m \times n$ matrix, then we can find an $m \times n$ matrix B such that

$$A + B = B + A = O_{m\times n}$$

The matrix B in above property is called 'additive inverse' or 'negative' of A and is denoted by − A.

Infact, if A = $[a_{ij}]_{m\times n}$ then –A = $[-a_{ij}]_{m\times n}$

Thus, property 4 can be written as

$$A + (-A) = (-A) + A = O_{m\times n}$$

We can now define difference of two Matrices.

Definition : Let A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{m\times n}$ two matrices. We define the **difference** A − B to be the $m \times n$ matrix A + (−B).
Note that A − B is of dimension $m\times n$ and $A - B = \left[a_{ij} - b_{ij}\right]_{m\times n.}$

For example, if A = $\begin{bmatrix} 4 & -1 & 6 \\ 5 & 8 & 3 \end{bmatrix}$ and B = $\begin{bmatrix} 2 & -5 & -3 \\ 7 & 0 & 8 \end{bmatrix}$

then A – B = $\begin{bmatrix} 4-2 & -1+5 & 6+3 \\ 5-7 & 8-0 & 3-8 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 9 \\ -2 & 8 & -5 \end{bmatrix}$.

Scalar Multiplication

Definition : Let A =$[a_{ij}]_{m\times n}$ be a matrix and let K be a complex number. The scalar multiplication KA of the matrix A and the number K (called the scalar) is the $m\times n$ matrix KA $[ka_{ij}]_{m\times n}$

For example, let $A = \begin{bmatrix} 3 & 1 \\ -2 & 0 \\ 5 & 1 \end{bmatrix}$

If K = 4, then kA = 4A = $\begin{bmatrix} 12 & 4 \\ -8 & 0 \\ 20 & 4 \end{bmatrix}$

and if $k = \frac{1}{3}$, then kA $= \frac{1}{3} A = \begin{bmatrix} 1 & 1/3 \\ -2/3 & 0 \\ 5/3 & 1/3 \end{bmatrix}$

Note that if $k = -1$, then (-1)A = $-$ A.

This is one of the properties of scalar multiplication. We list some of these properties without proof.

Properties of Scalar Multiplication

1. Let A = $[a_{ij}]_{m\times n}$ be a matrix and let k_1 and k_2 be two scalars. Then
   (i) $(k_1 + k_2)$ A = k_1A + k_2 A, and
   (ii) $k_1 (k_2$ A) = $(k_1 k_2)$A.

2. Let A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{m\times n}$ be two matrices and let k be a scalar. Then
   k (A + B) = kA + k B.

Multiplication of two Matrices

Let A = $[aij]_{m\times n}$ and B = $[bij]_{r\times s}$ be two matrices. We say that A and B are **comparable for the product** AB if $n = r$, that is, if the number of columns of A is same as the number of rows of B.

Definition : Let A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{n\times p}$ be two matrices. Their product AB is the matrix C = $[c_{ij}]_{m\times p}$ such that $c_{ij} = a_{i1} b_{ij} + a_{i2} b_{2j} + a_{i3} b_{3j} + \ldots\ldots + a_{in} b_{nj}$ for $i \le i \le m$, $1 \le j \le p$. Note that the order of AB is $m\times p$.

Example 2 : Let A = $\begin{bmatrix} 2 & 3 & 7 \\ -1 & 5 & 2 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 0 \\ 2 & 5 \\ -2 & 7 \end{bmatrix}$

Obtain the product AB.

Solution : Since A is of order 2×3 and B is or order 3×2, therefore, the product AB is defined. Order of AB is 2×2.

$$AB = \begin{bmatrix} 2 & 3 & 7 \\ -1 & 5 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 2 & 5 \\ -2 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2\times 3 + 3\times 2 + 7\times(-2) & 2\times 0 + 3\times 5 + 7\times 7 \\ (-1)\times 3 + 5\times 2 + 2\times(-2) & -1\times 0 + 5\times 5 + 2\times 7 \end{bmatrix}$$

$$= \begin{bmatrix} 6+6-14 & 0+15+49 \\ -3+10-4 & 0+25+14 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 64 \\ 3 & 39 \end{bmatrix}$$

Properties of Matrix Multiplication

Some of the properties satisfied by matrix multiplication are stated below without proof.

1. (Associative Law) : If A = $[a_{ij}]_{m\times n}$, B = $[b_{ij}]_{n\times p}$ and C = $[c_{ij}]_{p\times q}$ are three matrices, then

   (AB) C = A (BC).

2. (Distributive Law): If A = $[a_{ij}]_{m\times n}$, B = $[b_{ij}]_{n\times p}$ and C = $[c_{ij}]_{n\times p}$ are three matrices, then

   A (B + C) = AB + AC.

3. If A = $[a_{ij}]_{m\times n}$ and B = $[b_{ij}]_{n\times p}$ are two matrices, and k is a complex number, then

   (kA) B = A(kB) = k(AB).

4. If A = $[a_{ij}]_{m\times n}$ is an $m\times n$ matrix, then

   $I_m A = AI_n = A$,

   Where I_m and I_n are unit matrices of order m and n respectively.

Example 3: Let A = $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ and B = $\begin{bmatrix} 3 & 5 & -2 \end{bmatrix}$

Find AB and BA.

Solution : Since A is 3×1 matrix and B is a 1×3 matrix, therefore, AB is defined and its order is 3× 3.

If the number of columns of A is equal to the number of rows of B.

$$AB = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 3 & 5 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \times 3 & 1 \times 5 & 1 \times (-2) \\ 2 \times 3 & 2 \times 5 & 2 \times (-2) \\ (-1) \times 3 & (-1) \times 5 & -1 \times (-2) \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 & -2 \\ 6 & 10 & -4 \\ -3 & -5 & 2 \end{bmatrix}$$

Also, BA is defined and a is 1× 1 matrix

$$BA = \begin{bmatrix} 3 & 5 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

$$= [3 + 10 + 2] = [15]$$

This example illustrates that the matrix multiplication is not commutative. Infact, it may happen that the product AB is defined but BA is not, as in the following case :

$A = \begin{bmatrix} 1 & 2 \\ 5 & -3 \\ 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 1 & -1 \end{bmatrix}$

We now point out two more matrix properties which run counter to our experience to number systems.

1. It is possible that for two non-zero matrices and A and B, the product AB is a zero matrix.
2. It is possible that for a non-zero matrix A, and two unequal matrices B and C, we have, AB = AC. That is AB = AC, A ≠ 0 may not imply B = C. In other words, cancellation during multiplication does not hold.

These properties can be seen in the following example.

Example 4 : Let $A = \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 6 & 7 \\ 0 & 0 \end{bmatrix}$.

Show that $AB = O_{2\times 2}$ and $AB = AC$.

Solution : We have

$$AB= \begin{bmatrix} 0 & 3 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \text{ and}$$
$$= O_{2\times 2}$$

$$AC= \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 6 & 7 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0+0 & 0+0 \\ 0+0 & 0+0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$= O_{2\times 2}$$

Therefore, AB = AC. We see however, that $A \neq O_{2\times 2}$ and $B \neq C$. Thus, cancellation during multiplication does not hold.

Exponent of a Square Matrix

We now introduce the notion of the exponent of a square matrix. To begin with, we define A^m for any square matrix and for any positive integer *m*.

Let A be a square matrix and *m* a positive integer. We define.

$$A^m = \underbrace{AAA \ldots\ldots A}_{m \text{ times}}$$

More formally, the two equations A' = A and $A^{m+1} = A^m$ A define A^m recursively by defining it first for $m = 1$ and then $m+1$ after it has been defined for *m*, for all $m \geq 1$.

We also define $A° = I_n$, where A is a non−zero square matrix of order *n*.

The usual rules of exponent's namely

$A^m A^n = A^{m+n}$ and $(A^m)^n = A^{mn}$ do hold for matrices if *m* and *n* are non-negative integers.

Example 5 : Let $A = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$ and $f(x) = x^2 - 4x + 7$. Show that $f(A) = O_{2\times 2}$. Use this result to find A^5.

Solution : First, we note that by *f*(A) we mean $A^2 - 4A + 7I_2$. That is, we replace *x* by A and multiply the constant term by I, the unit matrix. Therefore,

$$f(A) = A^2 - 4A + 7I_2$$

$$= \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} + 7 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4-3 & 6+6 \\ -2-2 & -3+4 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 12 \\ -4 & 1 \end{bmatrix} - \begin{bmatrix} 8 & 12 \\ -4 & 8 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 1-8+7 & 12-12+0 \\ -4+4+0 & 1-8+7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = O_{2\times 2}.$$

Hence, $A^2 - 4A - 7\,I_2$, from which we get

$$A^3 = A^2A = (4A - 7I_2)A$$

$$= 4A^2 - 7I_2A = 4(4A - 7I_2) - 7A \quad [\because I_2\,A = A]$$

$$= 9A - 28\,I_2$$

$$\Rightarrow A^5 = A^2\,A^3 = (4A - 7I_2)\,(9A - 28I_2)$$

$$= 36A^2 - 63I_2A - 112AI_2 + 196\,I_2\,I_2 \quad \text{(Distributive Law)}$$

$$= 36\,(4A - 7I_2) - 63A - 112A + 196\,I_2$$

$$= 144A - 252\,I_2 - 175\,A + 196\,I_2$$

$$= -31A - 56\,I_2$$

$$= -31\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix} - 56\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -62 & -93 \\ 31 & -62 \end{bmatrix} - \begin{bmatrix} 56 & 0 \\ 0 & 56 \end{bmatrix}$$

$$= \begin{bmatrix} -118 & -93 \\ 31 & -118 \end{bmatrix}$$

Check Your Progress – 2

1. If P = $\begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix}$, Q = $\begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix}$, find matrix R such that 5P + 3Q + 2R is a null matrix.

2. If A = $\begin{bmatrix} 1 & -2 \\ 2 & -1 \end{bmatrix}$, B = $\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A+B)^2 = A^2 + B^2$ find a and b.

3. If A = $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ where $i^2 = -1$ verify $(A+B)^2 = A^2 + B^2$.

4. Let $f(x) = x^2 - 5x + 6$. Find $f(A)$ *if*

   A = $\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$.

5. If A = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, show that $A^5 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$.

6. If A and B are square matrices of the same order, explain why the following may not hold good in general.
   (a) $(A + B)(A - B) = A^2 - B^2$
   (b) $(A+B)^2 = A^2 + 2AB + B^2$
   (c) $(A-B)^2 = A^2 - 2AB + B^2$.

2.4 INVERTIBLE MATRICES

In this section, we restrict our attention to square matrices and formulate the notion of multiplicative inverse of a matrix.

Definition : An $n \times n$ matrix A is said to be **invertible** or **non-singular** if these exists an $n \times n$ matrix or **non singular** if there exists an $n \times n$ matrix such that AB = BA = I_n.

The matrix B is called an **inverse** of A. If there exists no such matrix B, then A is called **non-invertible** or **singular.**

Example 6 : Find whether A is invertible or not where

(a) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ (b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution : (a) We are asked whether we can find a matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that AB = I_2. What we require is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = AB = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$$

This would imply that $c = 0$, $d = 1$, $a = 1$ and $b = -1$, so that matrix

$$B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

does satisfy AB = I_2. Moreover, it also satisfies the equation BA = I_2. This can be verified as follows :

$$BA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & 1-1 \\ 0+0 & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

This implies that A is invertible and $B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is an inverse of A.

(b) Again we ask whether we can find a matrix $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that AB = I_2. What is required in this case is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}.$$

This would imply that a=1, b = 0 and the absurdity that 1=0. So no such B exists for this particular A. Hence, A is non invertible.

We will not show that if A is invertible, then B in the above definition is unique.

Theorem : If a matrix has an inverse, then inverse is unique.

Proof : Let B and C be inverses of a matrix A. Then by definition.

$$AB = BA = I_n \quad ..(1)$$

and $$AC = CA = I_n \quad .. (2)$$

Now, $B = BI_n$ [property of identify matrix]

$= B(AC)$ [(using 2)]

$= (BA)C$ [associative law]

$= I_nC$ [using (1)]

$= C.$ [property of identity matrix]

This means that we will always get the same inverse irrespective of the method employed. We will write the inverse of A, if it exists, as A^{-1}. Thus

$$AA^{-1} + A^{-1} A = I_n.$$

Definition

Let $A = (a_{ij})_{n\times n}$ be a square matrix of dimension $n\times n$. The cofactors matrix of A is defined to be the matrix $C = (a_{ij})_{n\times n}$ where A_{ij} denotes the cofactor of the element a_{ij} in the matrix A.

For example, if $A= \begin{pmatrix} 1 & -2 & 1 \\ 3 & 0 & 5 \\ 4 & -1 & 2 \end{pmatrix},$

then $A_{11=} (-1)^{1+1}\begin{vmatrix} 0 & 5 \\ -1 & 2 \end{vmatrix} = 5$

$A_{12=} (-1)^{1+2}\begin{vmatrix} 3 & 5 \\ 4 & 2 \end{vmatrix} = 14$ and

$A_{13=} (-1)^{1+3}\begin{vmatrix} 3 & 0 \\ 4 & -1 \end{vmatrix} = -3.$

Similarly, $A_{21} = 3$, $A_{22}= -2$ $A_{23} = -7$, $A_{31} = -10$, $A_{32} =-2$ and $A_{33} = 6$.

Thus, the cofactor matrix of A is given by $C = \begin{pmatrix} 5 & 14 & -3 \\ 3 & -2 & -7 \\ -10 & -2 & 6 \end{pmatrix}.$

Definition

The adjoint of square matrix $A = (a_{ij})_{n\times n}$ is defined to be the transpose of the cofactor matrix of A. It is denoted by *adj* A.

For example A = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then adj A = $\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix}$

The following theorem will enable us to calculate the inverse of a square matrix. We state the theorem (without proof) for 3×3 matrices only, but it is true for all square matrices of order $n \times n$, where $n \geq 2$.

Theorem : If A is a square matrix of order 3×3, then

A(adj)A = (adjA) A = $|A| I_3$.

In view of this theorem, we note that if $|A| \neq 0$, then

$A\left(\frac{1}{|A|}\text{adj } A\right) = \left(\frac{1}{|A|}\text{adj } A\right)A = I_3$.

Since, the inverse of a square matrix is unique, we see that if $|A| \neq 0$, then

$A\left(\frac{1}{|A|}\text{adj } A\right)$ acts as the inverse of A. That is,

$$\boxed{A^{-1} = \frac{1}{|A|}(\text{adjA})}$$

Also, a square matrix is invertible (non-singular) if and only if $|A| \neq 0$.

Example 7 : Find the inverse of A = $\begin{bmatrix} -3 & 5 \\ 2 & 4 \end{bmatrix}$

Solution :

We have $A_{11} = (-1)^{1+1}\ |4| = 4$ and $A_{12} = (-1)^{1+2}\ |2| = 2$.

We know that $|A| = a_{11}A_{11} + a_{12}A_{12} = (-3)(4) + 5(-2) = -22$.

Since $|A| \neq 0$ the matrix A is invertible, Also,
$A_{21} = (-1)^{2+1}\ |5| = -5$ and $A_{22} = (-1)^{2+2}\ |-3| = -3$. Therefore,

adj A = $\begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} 4 & -5 \\ -2 & -3 \end{pmatrix}$

Hence $A^{-1} = \frac{1}{|A|}$ adj A = $\frac{1}{-22}\begin{pmatrix} 4 & -5 \\ -2 & -3 \end{pmatrix} = \begin{pmatrix} -2/11 & 5/22 \\ 1/11 & 3/22 \end{pmatrix}$

Example 8 : If A = $\begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 4 & 5 \\ 2 & 5 \end{bmatrix}$,

verify that $(AB)^{-1} = B^{-1}A^{-1}$.

Solution : Since $|A| = -8 \neq 0$, $\therefore$ A is invertible.

Similarly, $|B| = 20 - 10 = 10 \neq 0$, $\therefore$ B is also invertible.

Let A_{ij} denote the cofactor of a_{ij} – the $(i,j)^{th}$ element of A. Then

$A_{11} = 0$, $A_{12} = -4$, $A_{21} = -2$ and $A_{22} = 3$.

Similarly, if B_{ij} is cofactor of $(i,j)^{th}$ element of B, then

$B_{11} = 5$, $B_{12} = -2$, $B_{21} = -5$ and $B_{22} = 4$

$\therefore$ adj A $= \begin{bmatrix} 0 & -2 \\ -4 & 3 \end{bmatrix}$ and adj B $= \begin{bmatrix} 5 & -5 \\ -2 & 4 \end{bmatrix}$

$\Rightarrow A^{-1} = \frac{1}{|A|}$ adj A $= \frac{-1}{8}\begin{bmatrix} 0 & -2 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1/2 & -3/8 \end{bmatrix}$

and $B^{-1} = \frac{1}{|B|}$ adj B $= \frac{1}{10}\begin{bmatrix} 5 & -5 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1/2 & -1/2 \\ -1/5 & 2/5 \end{bmatrix}$

Let $C = AB = \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 12+4 & 15+10 \\ 16+0 & 20+10 \end{bmatrix} = \begin{bmatrix} 16 & 25 \\ 16 & 20 \end{bmatrix}$

We have

$C_{11} = 20$, $C_{12} = -16$, $C_{21} = -25$ and $C_{22} = 16$

Also, $|C| = -80 \neq 0$, $\therefore$ C is invertible.

Also, adj C $= \begin{bmatrix} 20 & -25 \\ -16 & 16 \end{bmatrix}$

$$\Rightarrow C^{-1} = \frac{1}{|C|} \text{ adj C } = -\frac{1}{80}\begin{bmatrix} 20 & -25 \\ -16 & 16 \end{bmatrix} = \begin{bmatrix} -1/4 & 5/16 \\ 1/5 & -1/5 \end{bmatrix}.$$

Hence, $B^{-1}A^{-1} = \begin{bmatrix} 1/2 & -1/2 \\ -1/5 & 2/5 \end{bmatrix}\begin{bmatrix} 0 & 1/4 \\ 1/2 & -3/8 \end{bmatrix}$

$= \begin{bmatrix} -1/4 & 5/16 \\ 1/5 & -1/5 \end{bmatrix} = C^{-1} = (AB)^{-1}$

Example 9 : Find the inverse of $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$

and verify that $A^{-1}A = I_3$.

Solution : Evaluating the cofactors of the elements in the first row of A, we get

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2, \quad A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = -3,$$

and $A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 3 \\ -1 & 1 \end{vmatrix} = 5,$

$$\therefore \quad |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$= (1)(2) + (2)(-3) + (5)(5) = 21$$

Since $|A| \neq 0$, A is invertible. Also,

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} = 3, \qquad A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 5 \\ -1 & 1 \end{vmatrix} = 6,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -3, \qquad A_{31} = (-1)^{3+1}\begin{vmatrix} 2 & 5 \\ 3 & 1 \end{vmatrix} = -13,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 5 \\ 2 & 1 \end{vmatrix} = 9, \qquad A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = -1,$$

$$\therefore \text{ adj } A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{ adj}A = \frac{1}{21}\begin{bmatrix} 2 & 3 & -13 \\ -3 & 6 & 9 \\ 5 & -3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}$$

To verify that this is the inverse of A, we have

$$A^{-1}A = \begin{bmatrix} 2/21 & 3/21 & -13/21 \\ -3/21 & 6/21 & 9/21 \\ 5/21 & -3/21 & -1/21 \end{bmatrix}\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{21}+\frac{6}{21}+\frac{13}{21} & \frac{4}{21}+\frac{9}{21}+\frac{-13}{21} & \frac{10}{21}+\frac{3}{21}+\frac{-13}{21} \\ \frac{-3}{21}+\frac{12}{21}+\frac{9}{21} & \frac{-6}{21}+\frac{18}{21}+\frac{9}{21} & \frac{-15}{21}+\frac{6}{21}+\frac{19}{21} \\ \frac{5}{21}-\frac{6}{21}+\frac{1}{21} & \frac{10}{21}-\frac{9}{21}-\frac{1}{21} & \frac{25}{21}-\frac{3}{21}-\frac{1}{21} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Check Your Progress – 3

1. Find the adjoint of each of the following Matrices :

   (i) $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -1 & 3 & 5 \end{bmatrix}$

2. For A = $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 0 & 5 \\ 4 & -1 & 2 \end{bmatrix}$, verify that

   A (adj A) = (adj.A) A = |A| I_3.

3. Find the inverse of A = $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -1 & 3 & 5 \end{bmatrix}$

4. Let A = $\begin{bmatrix} 3 & 1 \\ 4 & 0 \end{bmatrix}$ and B = $\begin{bmatrix} 4 & 0 \\ 2 & 5 \end{bmatrix}$, Verify that

   $(AB)^{-1} = B^{-1}A^{-1}$.

5. If A = $\begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that $A^2 = A^{-1}$.

   What is Adj A ?

6. Let A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ Prove that $A^2 - 4A - 5I_3 = 0$.

   Hence, obtain A^{-1}

7. Find the condition under which

   A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible. Also obtain the inverse of A.

2.5 SYSTEMS OF LINEAR EQUATIONS

We can use matrices to solve a system of linear equations. Let us consider the following m linear equations in n unknowns :

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots . a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots . a_{2n}x_n = b_2$$

.

.

.

. (1)

$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots\ldots\ldots + a_{mn}x_n = b_m$$

where $b_1, b_2 \ldots\ldots b_m$ are not all zero.

The $m \times n$ matrix $\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$ is called the **coefficient matrix** of the system of linear equations. Using it, we can now write these equations as follows :

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ . \\ b_m \end{pmatrix}$$

We can abbreviate the above matrix equation to $AX = B$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

and X and B are the $n\times 1$ column vectors.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_n \end{pmatrix} \text{ and } B = \begin{pmatrix} b_1 \\ b_2 \\ . \\ b_m \end{pmatrix}$$

Recall that by a solution of (1) we mean a set of values $x_1.x_2\dots\dots\dots.x_n$ which satisfy all the equations in (1) simultaneously.

For example, $x_1 = 2$, $x_2 = -1$ is a solution of the system of linear equations.

$$3x_1 - 5x_2 = 11$$
$$2x_1 + 3x_2 = 1$$

because 3(2) – 5(–1) = 11 and 2(2) +3(–1) = 1.

Also, recall that the system of linear equations (1) is said to be consistent if it has at least one solution; it is *inconsistent* if it has no solution.

For example, the system of linear equations

$$\begin{aligned} 3x + 2y &= 5 \\ 6x + 4y &= 10 \end{aligned} \qquad (2)$$

is consistent. In fact, $x = k$, $y = \frac{1}{2}(5-2k)$ ($k \in C$) satisfies (2) for all values of $k \in C$. However, the system of linear equations.

$$\begin{aligned} 3x + 2y &= 5 \\ 6x + 4y &= 11 \end{aligned} \qquad (3)$$

is inconsistent. If this system has a solution $x = x_0$, $y = y_0$, then $3x_0 + 2y_0 = 5$ and $6x_0 + 4y_0 = 11$. Multiplying the first equation by 2 and subtracting from the second equation, we get 0 =1, which is not possible. Thus, the system in (3) has no solution and hence is an consistent.

Solution of AX= B (A non-singular)

Let us consider the system of linear equations $AX = B$, where A is an $n\times n$ matrix. Suppose that A is non-singular. Then A^{-1} exists and we can pre multiply $AX = B$ by A^{-1} on both sides to obtain

$$A^{-1}(AX) = A^{-1}(B)$$

$$\Rightarrow (A^{-1}A)X = A^{-1}B \qquad \text{[associative law]}$$

$$\Rightarrow I_nX = A^{-1}B \qquad \text{[property of law]}$$

$$\Rightarrow X = A^{-1}B \qquad \text{[property of identity matrix]}$$

Moreover, we have

$$A(A^{-1}B) = (AA^{-1})B \qquad \text{[associative law]}$$

$$= I_nB \qquad \text{[property of inverse]}$$

$$= B.$$

That is $A^{-1}B$ is a solution of $AX = B$. Thus, if A is non-singular, the system of equations $AX = B$ has a unique solution. This unique solution is given by $X = A^{-1}B$.

Example 10 : Solve the following system of equations by the matrix inverse method :

$$x + 2y = 4, \quad 2x + 5y = 9.$$

Solution : We can put the given system of equations into matrix notation as follows :

$$\begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 \\ 9 \end{pmatrix}.$$

Here the coefficient matrix is give by $A = \begin{pmatrix} 1 & 2 \\ 2 & 5 \end{pmatrix}$.

To check if A^{-1} exists, we note that $A_{11} = (-1)^{1+1}|5| = 5$ and $A_{12} = (-1)^{1+2}|2| = -2$.

$\therefore$ $|A| = a_{11}A_{11} + a_{12}A_{12} = (1)(5) + (2)(-2) = 1 \neq 0$.

Since $|A| \neq 0$ A is non- singular (invertible). We also have $A_{21} = (-1)^{2+1}|2| = -2$. $A_{22} = (-1)^{2+2}|1| = 1$. Therefore, the adjoint of A is

$$\text{adj } A = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|}\text{adj } A = \frac{1}{1}\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$$

$$\therefore X = A^{-1}B = \begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 20 - 18 \\ -8 + 9 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \text{ or } x = 2, \; y = 1.$$

Example 11 : Solve the following system of equations by using matrix inverse :

$$3x + 4y + 7z = 14, \quad 2x - y + 3z = 4, \quad 2x + 2y - 3z = 0$$

Solution : We can put the given system of equations into the single matrix equation $AX = B$, where

$$A = \begin{pmatrix} 3 & 4 & 7 \\ 2 & -1 & 3 \\ 1 & 2 & -3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 14 \\ 4 \\ 0 \end{pmatrix}$$

The cofactors of |A| are

$$A_{11} = (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ 2 & -3 \end{vmatrix} = -3 \qquad A_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 1 & -3 \end{vmatrix} = 9$$

and $A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} = 5.$

$\therefore \ |A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13} = (3)(-3) + 4(9) + 7(5) = 62.$

Since $|A| \neq 0$, A is non−singular (invertible). Its remaining cofactors are

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & 7 \\ 2 & -3 \end{vmatrix} = 26, \qquad A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 7 \\ 1 & -3 \end{vmatrix} = -16,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & 4 \\ 1 & 2 \end{vmatrix} = -2, \qquad A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 7 \\ -1 & -3 \end{vmatrix} = 19,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 7 \\ 2 & 3 \end{vmatrix} = 5, \qquad A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & 4 \\ 2 & -1 \end{vmatrix} = -11.$$

The adjoint of matrix A is given by

$$\text{adj } A = \begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}$$

$$\therefore \ A^{-1} = \frac{1}{|A|}\text{adj}A = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}$$

$$\text{Also}, X = A^{-1}B = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}\begin{pmatrix} 14 \\ 4 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{62}\begin{pmatrix} -3 & 26 & 19 \\ 9 & -16 & 5 \\ 5 & -2 & -11 \end{pmatrix}\begin{pmatrix} 14 \\ 4 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{62}\begin{pmatrix} -42 + 104 \\ 126 - 64 \\ 70 - 8 \end{pmatrix} = \frac{1}{62}\begin{pmatrix} 62 \\ 62 \\ 62 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Hence x =1, y = 1, z = 1 is the required solution.

Example 12 : If $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$

are two square matrices, verify that AB = BA = $6I_3$. Hence, solve the system of linear equations : $x - y = 3$, $2x+3y+4z = 17$, $y + 2z = 7$.

Solution :

$$AB = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 2+4+0 & 2-2+0 & -4+4+0 \\ 4-12+8 & 4+6-4 & -8-12+20 \\ 0-4+4 & 0+2+2 & 0-4+10 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 6I_3$$

and $$BA = \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix}\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2+4+0 & -2+6-4 & 0+8-8 \\ -4+4+0 & 4+6-4 & 0+8-8 \\ 2-2+0 & -2-3+5 & 0-4+10 \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{pmatrix} = 6\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 6I_3$$

Thus, AB =BA = $6I_3$

$$\Rightarrow \quad A\left(\frac{1}{6}B\right) = \left(\frac{1}{6}B\right)A = I_3$$

This shows that $A^{-1} = \frac{1}{6}B$. Now the given system of equations can be written as

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$$

or AX = C, where

$\therefore$ $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $C = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$

$$X = A^{-1} C = \frac{1}{6} BC \qquad \left[\because A^{-1} = \frac{1}{6} B\right]$$

$$= \frac{1}{6}\begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & 1 & 5 \end{pmatrix}\begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} 6 & 34 & -28 \\ -12 & +34 & -28 \\ 2 & -17 & +35 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 12 \\ -6 \\ 24 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}$$

Thus, $x = 2$, $y = -1$, $z = 4$ is the required solution.

Solution of a system of Homogeneous Linear Equations :

These are equations of the type $AX = O$. Let us consider the system of n homogeneous equations in n unknowns

$a_{11}\, x_1 + a_{12}\, x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{1n} x_n = 0$

$a_{21}\, x_1 + a_{22}\, x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{2n} x_n = 0$

$\vdots$

$a_{n1}\, x_1 + a_{n2}\, x_2 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + a_{nn}\, x_n = 0$

We can write this system as follows

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_n \\ a_{12} & a_{22} & \ldots & a_{2n} \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

We now abbreviate the above matrix equation to AX = 0, where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_n \\ a_{21} & a_{22} & \ldots & a_{2n} \\ & & & \cdot \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}$$

and X and O are the $n \times 1$ column vectors $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{pmatrix}$ and $O = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$.

If A is non-singular, then pre multiplying AX = O by A^{-1}, we get

$$A^{-1}(AX) = A^{-1}O$$

$\Rightarrow$ $(A^{-1}A)X = O$ [associative law]

$\Rightarrow$ $I_n X = O$ [property of inverse]

$\Rightarrow$ $X = O$ [property of identity matrix]

$\Rightarrow$ $x_1 = 0, x_2 = 0, \ldots\ldots\ldots x_n = 0.$

Also, note that $x_1 = 0$, $x_2 = 0$, ………. , $x_n = 0$ clearly satisfy the given system of homogeneous equations.

Thus, when A is non-singular AX = O has the unique solution. $x_1 = 0$, $x_2 = 0$, ………. $x_n = 0$. This is called the ***trivial solution.***

Important Result

We now state the following results without proof :

1. If A is singular, then AX = O has an infinite number of solutions.
2. Conversely, if AX = 0 has an infinite number of solution, then A is a singular matrix.

Example 13 : Solve the following system of homogeneous linear equations by the matrix method :

$$2x - y + z = 0, \quad 3x + 2y - z = 0, \; x + 4y + 3z = 0$$

Solution :

We can rewrite the above system of equations as the single matrix equation AX =0, where

$$A = \begin{pmatrix} 2 & -1 & 1 \\ 3 & 2 & -1 \\ 1 & 4 & 3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

The cofactors of |A| are

$$A_{11=} (-1)^{1+1} \begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix} = 10$$

$$A_{12=} (-1)^{1+2} \begin{vmatrix} 3 & -1 \\ 1 & 3 \end{vmatrix} = -10$$

and $A_{13=} (-1)^{1+3} \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix} = 10.$

$\therefore$ $|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (2)(10) + (-1)(-10) + 1(10) = 40.$

Since $|A| \neq 0, A$ is non $-$ singular (invertible). This, by known result $X = O$, that $x = 0, y = 0, z = 0.$

Example 14 : Solve the following system of homogeneous linear equation by the matrix method :

$$2x - y + 2z = 0, \quad 5x + 3y - z = 0, \quad x + 5y - 5z = 0$$

Solution :

We can rewrite the above system of equations as the single matrix equation AX =0, where

$$A = \begin{pmatrix} 2 & -1 & -2 \\ 5 & 3 & -1 \\ 1 & 5 & -5 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } O = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The cofactors of |A| are

$$A_{11=} (-1)^{1+1} \begin{vmatrix} 3 & -1 \\ 5 & -5 \end{vmatrix} = -10$$

$$A_{12=} (-1)^{1+2} \begin{vmatrix} 5 & -1 \\ 1 & -5 \end{vmatrix} = 24$$

$$\text{and} \quad A_{13=} (-1)^{1+3} \begin{vmatrix} 5 & 3 \\ 1 & 5 \end{vmatrix} = 22.$$

$$\therefore |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (2)(-10) + (-1)(24) + (2)(22) = 0.$$

Therefore, A is singular matrix. We can rewrite the first two equation as follows: $2x - y = -2z$, $5x + 3y = z$ or in the matrix form as

$$A = \begin{pmatrix} 2 & -1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -2z \\ z \end{bmatrix}.$$

Now, we have $A_{11} = (-1)^{1+1}|3| = 3$ and $A_{12} = (-1)^{1+2}|5| = -5$.

$$\therefore |A| = a_{11}A_{11} + a_{12}A_{12} = (2)(3) + (-1)(-5) = 11 \neq 0.$$

Thus, A is non singular (invertible). Also, $A_{21} = (-1)^{2+1}|-1| = 1$ and $A_{22} = (-1)^{2+2}|2| = 2$. Therefore, the adjoint of A is given by

$$\text{adj } A = \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|} \text{adj } A = \frac{1}{11} \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix}.$$

Therefore, from X = $A^{-1}B$, we get

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{11} \begin{pmatrix} 3 & 1 \\ -5 & 2 \end{pmatrix} \begin{pmatrix} -2z \\ z \end{pmatrix} = \frac{1}{11} \begin{pmatrix} -6z + z \\ 10z + 2z \end{pmatrix} = \begin{bmatrix} -\frac{5}{11}z \\ \frac{12}{11}z \end{bmatrix}$$

$$\Rightarrow x = -\frac{5}{11}z, \quad y = \frac{12}{11}z.$$

Let us check if these values satisfy the third equation. We have

$$x + 5y - 5z = -\frac{5}{11}z + 5\left(\frac{12}{11}z\right) - 5z = \frac{1}{11}z\,(-5z + 60z - 55z) = 0.$$

Thus, all the equation are satisfied by the values

$$\Rightarrow \quad x = -\frac{5}{11}z, \quad y = \frac{12}{11}z, \quad z = z.$$

Where z is any complex number. Hence, the given system of equation has an infinite number of solutions.

Solutions of AX = B (A Singular)

We state the following result without proof :

If A is a singular, that is |A| = 0, and

1. (adjA) B = 0, then AX = B has an infinite number of solutions (consistent).
2. (adj A) B $\neq 0$, then AX = B has no solution (inconsistent).

Example 15 : Solve the following system of linear equation by the matrix method :

$$2x - y + 3z = 5, \quad 3x + 2y - z = 7, \; 4x + 5y - 5z = 9$$

Solution :

We can rewrite the above system of equations as the single matrix equation AX =0, where

$$A = \begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & -1 \\ 4 & 5 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

Here, |A| = 0

$\therefore$ A is singular matrix. By calculating all the cofactors of A, we can write the adjoint of A. We have

$$adj\ A = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix}$$

$$\Rightarrow (adj\ A)B = \begin{bmatrix} -5 & 10 & -5 \\ 11 & -22 & 11 \\ 7 & -14 & 7 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus, AX = B has an infinite number of solutions. To find these solutions, we write $2x - y = 5 - 3z$, $3x + 2y = 7 + z$ or as a single matrix equation

$$\begin{bmatrix} 2 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 & -3z \\ 7 & +z \end{bmatrix}$$

Here, |A| = 7 $\neq$ 0

Since |A| $\neq$ 0, A is an invertible matrix

Now, adj A $= \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{7} \begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}$$

Therefore, from $X = A^{-1}B$, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 2 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} 5 & -3z \\ 7 & +z \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{17-5z}{7} \\ \dfrac{-1+11z}{7} \end{bmatrix}$$

$$\Rightarrow \; x = \frac{17-5z}{7}, \qquad y = \frac{1}{7}(-1+11z).$$

Let us check that these values satisfy the third equation. We have

$$4x + 5y - 5z = \frac{4}{7}(17 - 5z) + \frac{5}{7}(-1+11z) - 5z$$

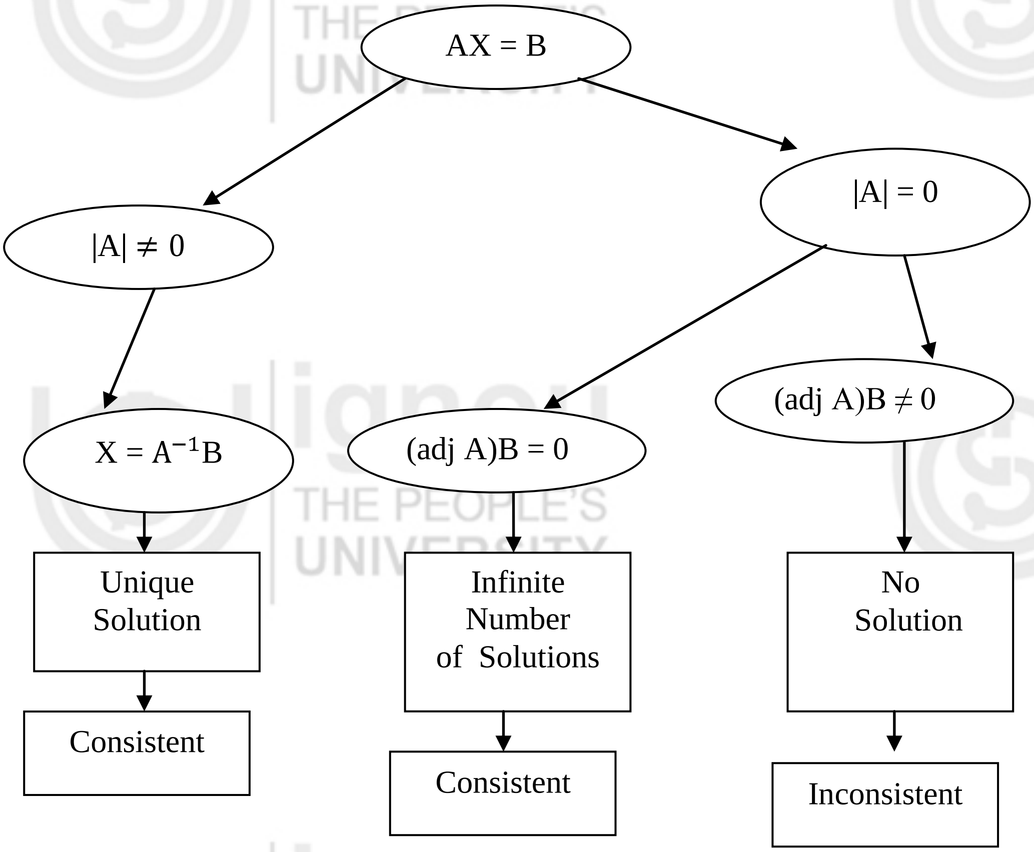
$$= \frac{1}{7}(68 - 20z - 5 + 55z - 35z) = 9.$$

Thus, the values

$$x = \frac{1}{7}(17-5z), y = \frac{1}{7}(-1+11z), \; z = z \quad (z \in C)$$

Satisfy, the given system which therefore has an infinite number of solutions.

In the end, we summarize the results of this section for a square matrix A in the form of a tree diagram.



Check Your Progress 4

1. Solve the following equations by matrix inverse method :
   $4x - 3y = 5,\ 3x - 5y = 1$
2. Use the matrix inverse to solve the following system of equations :
   (a) $x + y - z = 3,\ 2x + 3y + z = 10,\ 3x - y - 7z = 1$
   (b) $8x + 4y + 3z = 18,\ 2x + y + z = 5,\ x + 2y + z = 5$
3. Solve the following system of homogeneous linear equations by the matrix method :
   $3x - y + 2z = 0,\ 4x + 3y + 3z = 0,\ 5x + 7y + 4z = 0$
4. Solve the following system of linear equations by the matrix method :
   $3x + y - 2z = 7$
   $5x + 2y + 3z = 8$
   $8x + 3y + 8z = 11$

2.6 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1. Note that a 2×2 matrix is given by

   $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

   From the formulas given the elements, we have

   (a) $A = \begin{bmatrix} 9/2 & 25/2 \\ 8 & 18 \end{bmatrix}$

   (b) $A = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$

2. From equality of matrices, we have,
   $x = 2x + y,\ y = x - y$

   Solving we get $x = 0$, $y = 0$

3. We have
   $a - b = 5 \quad 2c + d = 3$
   $2a - b = 12 \quad 2a + d = 15$
   Solving we get $a = 7$, $b = 2$, $c = 1$ and $d = 1$.

4. (a) $A' = \begin{bmatrix} 2 & 2 \\ 2 & 6 \end{bmatrix} = A$

   $\therefore$ A is symmetric matrix

   (b) $A' = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} = -A$

   $\therefore$ A is skew − symmetric matrix

(c) $A' = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & -5 \\ 3 & 5 & 0 \end{bmatrix} = -A$

$\therefore$ A is skew $-$ symmetric matrix

Check Your Progress - 2

1. Since P and Q are matrices of order 2× 2, 5P + 3Q is a matrix of order 2×2 and therefore R must be a matrix or order 2 × 2.

Let R = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then

$5P + 3Q + 2R = 5\begin{bmatrix} 9 & 1 \\ 7 & 8 \end{bmatrix} + 3\begin{bmatrix} 1 & 5 \\ 7 & 12 \end{bmatrix} + 2\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$= \begin{bmatrix} 45 & 5 \\ 35 & 40 \end{bmatrix} + \begin{bmatrix} 3 & 15 \\ 21 & 36 \end{bmatrix} + \begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$

$= \begin{bmatrix} 48+2a & 20+2b \\ 56+2c & 76+2d \end{bmatrix}$

Since 5P + 3Q + 2 R = $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, we get

$48 + 2a = 0,\ 20 + 2b = 0,\ 56 + 2c = 0,\ 76 + 2d = 0$

$\Rightarrow \quad a = -24, b = -10, c = -28$ and $d = -36$.

Thus, R = $\begin{bmatrix} -24 & -10 \\ -28 & -36 \end{bmatrix}$

2. We have $(A+B)^2 = (A+B)(A+B)$

$= (A+B)A + (A+B)B$ (Distributive Law)

$= AA + BA + AB + BB$

$= A^2 + BA + AB + B^2$

Therefore, $(A+B)^2 = A^2 + B^2$

$\Rightarrow A^2 + BA + AB + B^2 = A^2 + B^2$

$\Rightarrow BA + AB = 0.$

Thus, we must find a and b such that BA + AB = 0.

We have BA = $\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix}$

and AB = $\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$

Therefore,

BA + AB = $\begin{bmatrix} a+2 & -a-1 \\ b-2 & -b+1 \end{bmatrix} + \begin{bmatrix} a-b & 2 \\ 2a-b & 3 \end{bmatrix}$

$$= \begin{bmatrix} 2a-b+2 & -a+1 \\ 2a-2 & -b+4 \end{bmatrix}$$

But BA + AB = 0

$\Rightarrow$ $2a - b + 2 = 0$, $-a + 1 = 0$, $2a - 2 = 0$, $-b + 4 = 0$

$\Rightarrow a = 1, b = 4$

3. In view of discussion in solution (2), it is sufficient to show that BA + AB = 0

We have $BA = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$

and $AB = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

Thus, $BA + AB = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} + \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

4. First, we note that by f(A) we mean

$A^2 - 5A + 6I_3$. We have

$$A^2 = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & 2 \end{bmatrix}$$

Therefore,

$$A^2 - 5A + 6I_3 = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & 2 \end{bmatrix} - 5\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$$

5. We have $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Therefore, $A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$A^4 = A^2\ A^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}$

and $A^5 = A^4\ A = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$

6. In general matrix multiplication is not commulative. Therefore, AB may not be equal to BA, even though both of them exist.

Check Your Progress – 3

1. (i) Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The cofactors are

$A_{11} = (-1)^{1+1}\ |d| = d$

$A_{12} = (-1)^{1+2}\ |c| = -c$

$A_{21} = (-1)^{1+2}\ |b| = -b$ and

$A_{22} = (-1)^{2+2}\ |a| = a$

$\therefore$ adj A = $\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$.

(ii) Let A = $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ -1 & 3 & 5 \end{bmatrix}$

The cofactors of the elements of A are

$A_{11} = (-1)^{1+1} \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} = -1,$ $\quad A_{12} = (-1)^{1+2} \begin{bmatrix} 0 & 2 \\ -1 & 5 \end{bmatrix} = -2,$

$A_{13} = (-1)^{1+3} \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix} = 1,$ $\quad A_{21} = (-1)^{2+1} \begin{bmatrix} -1 & 3 \\ 3 & 5 \end{bmatrix} = 14,$

$A_{22} = (-1)^{2+2} \begin{bmatrix} 2 & 3 \\ -1 & 5 \end{bmatrix} = 13,$ $\quad A_{23} = (-1)^{2+3} \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} = -5,$

$A_{31} = (-1)^{3+1} \begin{bmatrix} -1 & 3 \\ 1 & 2 \end{bmatrix} = -5,$ $\quad A_{32} = (-1)^{3+2} \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} = -4,$

$A_{33} = (-1)^{3+3} \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix} = 2.$

$\therefore$ adj A = $\begin{bmatrix} -1 & 14 & 5 \\ -2 & 13 & -4 \\ 1 & -5 & 2 \end{bmatrix}$

2. For the given matrix A, we have

adj A = $\begin{bmatrix} 5 & 3 & -10 \\ 14 & 2 & -2 \\ -3 & -7 & 6 \end{bmatrix}$

Now, A (adj A) = $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 0 & 5 \\ 4 & -1 & 2 \end{bmatrix} \begin{bmatrix} 5 & 3 & -10 \\ 14 & 2 & -2 \\ -3 & -7 & 6 \end{bmatrix}$

$$= \begin{bmatrix} -26 & 0 & 0 \\ 0 & -26 & 0 \\ 0 & 0 & -26 \end{bmatrix} = -26 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Similarly, (adj A) A = $\begin{bmatrix} 5 & 3 & -10 \\ 14 & 13 & -2 \\ -3 & -7 & 6 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 3 & 0 & 5 \\ 4 & -1 & 2 \end{bmatrix}$

$$= -26\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Also, $|A| = -26$

So, A (adj A) = (adj A) A = $|A|\, I_3$.

3. Here, $|A| = (2)(-1) + (-1)(-2) + (3)(1) = 3$

and adj A = $\begin{bmatrix}-1 & 14 & 5\\-2 & 13 & -4\\1 & -5 & 2\end{bmatrix}$ (see solution 1(ii))

$\therefore\ A^{-1} = \dfrac{1}{|A|}$ adj A

$$= \frac{1}{3}\begin{bmatrix}-1 & 14 & 5\\-2 & 13 & -4\\1 & -5 & 2\end{bmatrix} = \begin{bmatrix}-1/3 & 14/3 & 5/3\\-2/3 & 13/3 & -4/3\\1/3 & -5/3 & 2/3\end{bmatrix}.$$

4. We have $|A| = -4$ and $|B| = 20$. So, A and B are both invertible.
Also, adj A = $\begin{bmatrix}0 & -1\\-4 & 3\end{bmatrix}$ and adj B = $\begin{bmatrix}5 & 0\\-2 & 4\end{bmatrix}$.

$$\Rightarrow\ A^{-1} = \frac{1}{|A|}\text{adj } A = -\frac{1}{4}\begin{bmatrix}0 & -1\\-4 & 3\end{bmatrix} = \begin{bmatrix}0 & 1/4\\1 & -3/4\end{bmatrix}$$

and $B^{-1} = \dfrac{1}{|B|}\text{adj } B = \dfrac{1}{20}\begin{bmatrix}5 & 0\\-2 & 4\end{bmatrix} = \begin{bmatrix}1/4 & 0\\-1/10 & 1/5\end{bmatrix}$

Let C = AB = $\begin{bmatrix}3 & 1\\4 & 0\end{bmatrix}\begin{bmatrix}4 & 0\\2 & 5\end{bmatrix} = \begin{bmatrix}14 & 5\\16 & 0\end{bmatrix}$

So, $|C| = -80$ and adj C = $\begin{bmatrix}0 & -5\\-16 & 14\end{bmatrix}$

So, $C^{-1} = \dfrac{-1}{80}\begin{bmatrix}0 & -5\\-16 & 14\end{bmatrix} = \begin{bmatrix}0 & 1/16\\1/5 & -7/40\end{bmatrix}$

Hence, $B^{-1}A^{-1} = \begin{bmatrix}1/4 & 0\\-1/10 & 1/5\end{bmatrix}\begin{bmatrix}0 & 1/4\\1 & -3/4\end{bmatrix}$

$$= \begin{bmatrix}0 & 1/16\\1/5 & -7/40\end{bmatrix} = C^{-1} = (AB)^{-1}$$

5. We have

$$A^2 = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

To show that $A^2 = A^{-1}$, it is enough to show that $A(A^2) = I_3$. We have

$$A(A^2) = \begin{bmatrix} -1 & 2 & 0 \\ -1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3.$$

From $A^{-1} = \frac{1}{|A|}$ adj A, we get adj A $= |A|A^{-1}$. To obtain $|A|$, observe that $A^3 = I_3$ that is, $|A|^3 = |I_3| = 1$ *or* $|A| = 1$. Therefore adj A $= A^{-1}$

6. We have

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

Therefore,

$$A^2 - 4A - 5I_3 = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Also, $|A| = 5 \neq 0$. Therefore, A is invertible pre− multiplying $A^2 - 4A - 5I_3 = 0$ by A^{-1}, we get

$$A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I_3 = 0$$

$$\Rightarrow A - 4I_3 - 5A^{-1} = 0$$

$$\therefore \quad 5A^{-1} = A - 4I_3 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} = \begin{bmatrix} -3/5 & 2/5 & 2/5 \\ 2/5 & -3/5 & 2/5 \\ 2/5 & 2/5 & -3/5 \end{bmatrix}$$

7. We have $|A| = ad-bc$. Recall that A is invertible if and only if $|A| \neq 0$. That is A $= \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible if and only if $ad - bc \neq 0$.

Also, adj A $= \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Hence $A^{-1} = \frac{1}{|A|}$ adj $A = \frac{1}{ad - bc}\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$

Check Your Progress – 4

1. We can put the given system of equations into the single matrix equation.

$$\begin{bmatrix}4 & -3\\ 3 & -5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}5\\ 1\end{bmatrix}.$$

Here the coefficient matrix is given by A = $\begin{bmatrix}4 & -3\\ 3 & -5\end{bmatrix}$

Cofactors of |A| are $A_{11} = (-1)^{1+1}(-5)$ and $A_{12} = (-1)^{1+2}|3| = -3.$

$\therefore$ |A| = $a_{11}\,A_{11} + a_{12}\,A_{12} = (4)(-5) + (-3)(-3) = -11$

Since |A| $\neq$ 0. A is non-singular (invertible). Also $A_{21} = (-1)^{2+1}|-3|$and

$A_{12} = (-1)^{2+2}|4|$ = 4.

$$\therefore \quad \text{adj A} = \begin{pmatrix}-5 & 3\\ -3 & 4\end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{-1}{11}\begin{pmatrix}-5 & 3\\ -3 & 4\end{pmatrix}$$

$$\therefore \quad X = A^{-1}B = \frac{-1}{11}\begin{pmatrix}-5 & 3\\ -3 & 4\end{pmatrix}\begin{bmatrix}5\\ 1\end{bmatrix} \Rightarrow \begin{bmatrix}x\\ y\end{bmatrix} = \frac{-1}{11}\begin{pmatrix}-25+3\\ -15+4\end{pmatrix} = \begin{bmatrix}2\\ 1\end{bmatrix}$$

Hence x = 2, y = 1 is the required solution.

2. (a) We can put the above system of equation into the single matrix equation AX = B, where

$$A = \begin{pmatrix}1 & 1 & -1\\ 2 & 3 & 1\\ 3 & -1 & -7\end{pmatrix},\ X = \begin{pmatrix}x\\ y\\ z\end{pmatrix} \text{ and } B = \begin{pmatrix}3\\ 10\\ 1\end{pmatrix}$$

The cofactors of |A| are

$$A_{11=}(-1)^{1+1}\begin{vmatrix}3 & 1\\ -1 & -7\end{vmatrix} = -20 \qquad A_{12=}(-1)^{1+2}\begin{vmatrix}2 & 1\\ 3 & -7\end{vmatrix} = 17$$

and $A_{13\,=}(-1)^{1+3}\begin{vmatrix}2 & 3\\ 3 & -1\end{vmatrix} = -11.$

$\therefore\ a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 1(-20) + (1)(17) + (-1)(-11) = 8.$

Since |A| $\neq$ 0, A is non- singular (invertible). The remaining cofactors are

$$A_{21=}(-1)^{2+1}\begin{vmatrix}1 & -1\\ -1 & 7\end{vmatrix} = 8 \quad A_{22=}(-1)^{2+2}\begin{vmatrix}1 & -1\\ 3 & -7\end{vmatrix} = -4,$$
$$A_{23=}(-1)^{2+3}\begin{vmatrix}1 & 1\\ 3 & -1\end{vmatrix} = 4 \quad A_{31=}(-1)^{3+1}\begin{vmatrix}1 & -1\\ 3 & 1\end{vmatrix} = 4,$$

$$A_{32=}(-1)^{3+2}\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix} = -3 \quad A_{33=}(-1)^{3+3}\begin{vmatrix}1 & 1\\ 2 & 3\end{vmatrix} = 1.$$

$$\therefore \text{ adj A } = \begin{pmatrix} -20 & 18 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{8} \begin{pmatrix} -20 & 8 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix}$$

$$\text{Also, X } = A^{-1}B = \frac{1}{8} \begin{pmatrix} -20 & 18 & 4 \\ 17 & -4 & -3 \\ -11 & 4 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 10 \\ 1 \end{pmatrix}$$

$$= \frac{1}{8} \begin{pmatrix} -60 + 80 + 4 \\ 51 - 40 - 3 \\ -33 + 40 + 1 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 24 \\ 8 \\ 8 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

Thus, $x = 3, y = 1, z = 1$ is the required solution.

2. (b) We can put the above system of equation into the single matrix equation AX = B, where

$$\text{A} = \begin{pmatrix} 8 & 4 & 3 \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{pmatrix}, \quad \text{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and B} = \begin{pmatrix} 18 \\ 5 \\ 5 \end{pmatrix}.$$

The cofactors of |A| are

$$A_{11=} (-1)^{1+1} \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1 \qquad A_{12=} (-1)^{1+2} \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = -1$$

$$\text{and} \quad A_{13=} (-1)^{1+3} \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3.$$

$$\therefore \; |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = 8(-1) + 4(-1) + 3(3) = -3.$$

Since |A| ≠ 0, A is non−singular (invertible). The remaining cofactors are

$$A_{21=} (-1)^{2+1} \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 2 \qquad A_{22=} (-1)^{2+2} \begin{vmatrix} 8 & 3 \\ 1 & 1 \end{vmatrix} = 5,$$

$$A_{23=} (-1)^{2+3} \begin{vmatrix} 8 & 4 \\ 1 & 2 \end{vmatrix} = -12 \qquad A_{31=} (-1)^{3+1} \begin{vmatrix} 4 & 3 \\ 1 & 1 \end{vmatrix} = 1,$$

$$A_{32=} (-1)^{3+2} \begin{vmatrix} 8 & 3 \\ 2 & 1 \end{vmatrix} = -2 \qquad A_{33=} (-1)^{3+3} \begin{vmatrix} 8 & 4 \\ 2 & 1 \end{vmatrix} = 0.$$

$$\therefore \text{ adj A } = \begin{pmatrix} -1 & 2 & 1 \\ -1 & 5 & -2 \\ 3 & -12 & 0 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \frac{1}{|A|} \text{ adj } A = \frac{1}{-3} \begin{pmatrix} -1 & 2 & 1 \\ -1 & 5 & -2 \\ 3 & -12 & 0 \end{pmatrix}.$$

$$\text{Also, } X = A^{-1}B = \frac{1}{-3}\begin{pmatrix}-1 & 2 & 1\\ -1 & 5 & -2\\ 3 & -12 & 0\end{pmatrix}\begin{pmatrix}18\\ 5\\ 5\end{pmatrix}$$

$$= \frac{1}{-3}\begin{pmatrix}-18+10+5\\ -18+25-10\\ 54-60+0\end{pmatrix} = \frac{1}{-3}\begin{pmatrix}-3\\ -3\\ -6\end{pmatrix} = \begin{pmatrix}1\\ 1\\ 2\end{pmatrix}.$$

Thus, $x = 1$, $y = 1$, $z = 2$ is the required solution.

3. We can put the above system of equation into the single matrix equation AX = 0, where

$$A = \begin{pmatrix}3 & -1 & 2\\ 4 & 3 & 3\\ 5 & 7 & 4\end{pmatrix}, \quad X = \begin{pmatrix}x\\ y\\ z\end{pmatrix} \text{ and } B = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}$$

The cofactors of |A| are

$$A_{11} = (-1)^{1+1}\begin{vmatrix}3 & 3\\ 7 & 4\end{vmatrix} = -9 \qquad A_{12} = (-1)^{1+2}\begin{vmatrix}4 & 3\\ 5 & 4\end{vmatrix} = -1$$

and $$A_{13} = (-1)^{1+3}\begin{vmatrix}4 & 3\\ 5 & 7\end{vmatrix} = 13.$$

$$\therefore \quad |A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13} = (3)(-9) + (-1)(-1) + (2)(13) = 0.$$

Therefore, A is a singular matrix. We can rewrite the first two equations as follows :

$3x - y = -2z$, $4x + 3y = -3z$

or in the matrix form as $\begin{pmatrix}3 & -1\\ 4 & 3\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}-2z\\ -3z\end{pmatrix}$.

Now, we have $A_{11} = (-1)^{1+1}|3| = 3$ and $A_{12} = (-1)^{1+2}|4| = -4$.

$$\therefore \quad |A| = a_{11}A_{11} + a_{12}A_{12} = (3)(3) + (-1)(-4) = 13 \neq 0.$$

Thus, A is non- singular (invertible).

Also, $A_{21} = (-1)^{2+1}|-1| = 1$ and $A_{22} = (-1)^{2+2}|3| = 3$.

$$\therefore \text{adj}A = \begin{pmatrix}3 & 1\\ -4 & 3\end{pmatrix} \Rightarrow A^{-1} = \frac{1}{|A|}\text{ adj } A = \frac{1}{13}\begin{pmatrix}3 & 1\\ -4 & 3\end{pmatrix}$$

Therefore, from $X = A^{-1}B$, we get

$$\begin{pmatrix}x\\ y\end{pmatrix} = \frac{1}{13}\begin{pmatrix}3 & 1\\ -4 & 3\end{pmatrix}\begin{pmatrix}-2z\\ -3z\end{pmatrix} = \frac{1}{13}\begin{pmatrix}-6z-3z\\ 8z-9z\end{pmatrix} = \begin{pmatrix}-\frac{9}{13}z\\ -\frac{1}{13}z\end{pmatrix}.$$

$$\Rightarrow x = -\frac{9}{13}z, \qquad y = -\frac{1}{13}z.$$

Let us check if these values satisfy the third equation. We have

$$5x + 7y + 4z = 5\left(-\frac{9}{13}z\right) + 7\left(-\frac{1}{13}z\right) + 4z$$
$$= \frac{z}{13}(-45 - 7 + 52) = \frac{0}{13}z = 0.$$

Thus, all the equations are satisfied by the values

$$x = -\frac{9}{13}z, \qquad y = -\frac{1}{13}z, \qquad z = z.$$

4. We can write the given system of linear equation as the single matrix equation.

AX = B,

Where

$$A = \begin{pmatrix} 3 & 1 & -2 \\ 5 & 2 & 3 \\ 7 & 3 & 8 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ and } B = \begin{pmatrix} 7 \\ 8 \\ 11 \end{pmatrix}$$

Here, |A| = 0

Therefore, A is a singular matrix.

$$\text{Now adj } A = \begin{pmatrix} 7 & -14 & 7 \\ -19 & 38 & -9 \\ 1 & -2 & 1 \end{pmatrix}$$

$$\Rightarrow (\text{adj } A)B = \begin{pmatrix} 7 & -14 & 7 \\ -19 & 38 & -19 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ 8 \\ 11 \end{pmatrix} = \begin{pmatrix} 14 \\ -38 \\ 2 \end{pmatrix}$$

Since (adj A) B ≠ 0, the given system of equations has no solution (inconsistent).

2.7 SUMMARY

In this unit, first of all, definition and notation of an m x n matrix, are given in **section 2.2**. Next, in this section, special types of matrices, viz., square matrix, diagonal matrix, scalar matrix, unit or identity matrix, row or column matrix and zero or null matrix are also defined. Then, equality of two matrices, transpose of a matrix, symmetric and skew matrices are defined. Each of the above concepts is explained with a suitable example. In **section 2.3**, operations like addition, subtraction, multiplication of two matrices and multiplication of a matrix with a scalar are defined. Further, properties of these matrix operations are stated without proof. Each of these operations is explained with a suitable example. In **section 2.4**, the concepts of an invertible matrix, cofactors of a matrix, adjoint of a square matrix are defined and explained with suitable examples. Finally, in **section 2.5**, method of solving linear equations in n variables using matrices, is given and illustrated with a number of suitable examples. Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 2.6**.