UNIT 3 EQUATIONS

Structure

3.0 INTRODUCTION

Solution of equations lies at the very heart of algebra. It has enormous applications. The importance of equations stems from the fact that they provide a means by which many complicated relationships in real–life problems can be written down in a clear and concise form.

In earlier classes you studied how to solve a first degree (linear) equation.

$bx + c = 0$ ($b \neq 0$).

Recall that you had obtained its roots as $x = -c/b$.

In this unit, we shall take up solving second, third and fourth degree equations in one variable.

3.1 OBJECTIVES

After studying this unit, you will be able to:

- solve a quadratic equation, cubic and biquadractic equations;
- find values of symmetric expressions involving roots of a cubic and quadratic equation;
- form equations whose roots are known; and
- use equations to solve several real life problems.

3.2 SOLUTION OF QUADRATIC EQUATIONS

Definition : Any equation that can be written in the form

$$ax^2 + bx + c = 0$$

where a, b and c are real number and a not equal to 0 ($a \neq 0$), is called a quadratic equation. The form $ax^2 + bx + c = 0$ is called the standard form for a quadratic equation.

3.3 QUADRATIC FORMULA

We now use the method of completing the square to obtain the formula for roots of a quadratic equation. Towards this end we first list the steps for completing the square.

Steps for completing the Square

Step 1 : Write an equivalent equation with only the x^2 term and the x term on the left side of the equation. The coefficient of the x^2 term must be 1.

Step 2 : Add the square of one–half the coefficient of the x term to both sides of the equation.

Step 3 : Express the left side of the equation as a perfect square.

Step 4 : Solve for x.

Theorem (The Quadratic Formula)

For any quadratic equation in the form $ax^2 + bx + c = 0$, where $a \neq 0$, the two solutions are

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Proof : We shall prove this quadratic theorem by completing the square on $ax^2 + bx + c = 0$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \qquad \text{[divide by } a\text{]}$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \left(\frac{1}{2}.\frac{b}{a}\right) = -\frac{c}{a} + \left(\frac{1}{2}.\frac{b}{a}\right) \quad [add \ \left(\frac{1}{2}.\frac{b}{a}\right) \text{ to each side]}$$

$$\Rightarrow \quad x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \qquad \left[x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2\right]$$

$$\Rightarrow x^2 + \frac{b}{a}x = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \qquad [\text{if } x^2 = k, x = \pm\sqrt{k}\,]$$

$$\Rightarrow \quad x + \frac{b}{2a} = \frac{\sqrt{b^2 - 4ac}}{2a} \text{ or } x + \frac{b}{2a} = -\frac{\sqrt{b^2 - 4ac}}{2a} \qquad [\text{solve for } x]$$

$$\Rightarrow x = \frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a} \text{ or } x = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

This complete the proof. Let us see what have proved. If our equation is in the form $ax^2 + bx + c = 0$(standard form), where $a \neq 0$, the two solutions are always given by the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

This formula is known as the *quadratic formula*. If we substitute the coefficient *a*, *b* and *c* of any quadratic equation in standard form in the formula, we need only perform some basic arithmetic to arrive at the solution set.

The Nature of Solutions

Since the solution set for every quadratic equation is

$$\left\{\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}\right\},$$

the solutions can be expressed as

$$\alpha = \frac{-b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \; or \; \beta = \frac{-b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Discriminant

The expression D = $b^2 - 4ac$ is called discriminant because it determines the nature of the solutions of a quadratic equation.

1. If $b^2 - 4ac = 0$ then $\alpha = \beta$ and the equation will have one real root.

2. If $b^2 - 4ac > 0$ then α and β will be two distinct real numbers, and the equation will have two, unequal real roots.

3. If $b^2 - 4ac < 0$ then α and β will be two distinct complex numbers, and the equation will have no real roots.

If D = $b^2 - 4ac < 0$ then $4ac - b^2 > 0$. In this case, the complex number $\omega_1 = i\sqrt{4ac - b^2}$ and $\omega_2 = -i\sqrt{4ac - b^2}$ are such that $\omega^2 = b^2 - 4ac$ and no other complex number z is such that $z^2 = 4ac - b^2$. In this case the two roots may written as

$$\alpha = \frac{-b + \omega_1}{2a}, \text{and} \quad \beta = \frac{-b - \omega_2}{2a}$$

Sum and Product of the roots

We have

$$\alpha + \beta = \frac{-b + \sqrt{b^2 - 4ac}}{2a} + \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}$$

$$\text{and } \alpha\beta = \left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)\left(\frac{-b - \sqrt{b^2 - 4ac}}{2a}\right)$$

$$= \left(\frac{-b}{2a}\right)^2 - \left(\frac{\sqrt{b^2 - 4ac}}{2a}\right)^2 \qquad [(a+b)(a-b) = a^2 - b^2]$$

$$= \frac{b^2}{4a^2} - \frac{b^2 - 4ac}{4a^2} = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}$$

Thus, if α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = \frac{-b}{a} \text{ and } \alpha\beta = \frac{c}{a}.$$

Forming Quadratic Equations with given Roots

An equation whose roots are α and β can be written as $(x - \alpha)(x - \beta) = 0$

or $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

or x^2 – (Sum of roots) x + (Product of the roots) = 0

Example 1: If α and β are the roots of the equation $ax^2 + bx + c = 0$, $a \neq 0$ find the value of

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$ (ii) $\alpha^2 + \beta^2$ (iii) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (iv) $\alpha^3 + \beta^3$

(v) $\alpha^2 + \beta^2$ (vi) $\alpha^{-3} + \beta^{-3}$ (vii) $\left(\frac{\alpha}{\beta} - \frac{\beta}{\alpha}\right)^2$ (viii) $\alpha^2 - \beta^2$

Solution:

Since α and β are the roots of $ax^2 + bx + c = 0$, $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$.

$$(i) \quad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{-b/a}{c/a} = \frac{-b}{c}.$$

$$(ii) \quad \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(\frac{-b}{a}\right)^2 - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}.$$

$$(iii) \quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \left(\frac{b^2 - 2ac}{a^2}\right)\left(\frac{a}{c}\right) = \frac{b^2 - 2ac}{ac}. \qquad \text{[see(ii)]}$$

$$(iv) \quad \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) \qquad [a^3 + b^3 = (a + b)^3 - 3ab(a + b)]$$

$$= \left(\frac{-b}{a}\right)^3 - \frac{3c}{a}\left(\frac{-b}{a}\right) = \frac{-b^3}{a^3} + \frac{3bc}{a^3} = \frac{3abc - b^3}{a^3}.$$

$$(v) \quad \alpha^6 + \beta^6 = (\alpha^3)^2 + (\beta^3)^2 = (\alpha^2 + \beta^2)^3 - 3\alpha^2\beta^2(\alpha^2 + \beta^2)$$

$$[a^3 + b^3 = (a + b)^3 - 3ab(a + b)]$$

$$= [(\alpha + \beta)^2 - 2\alpha\beta]^3 - 3(\alpha\beta)^2[(\alpha + \beta)^2 - 2\alpha\beta]$$

$$= \left[\left(\frac{-b}{a}\right)^2 - \frac{2c}{a}\right]^2 - 3\left(\frac{c}{a}\right)^2\left[\left(\frac{-b}{a}\right)^2 - \frac{2c}{a}\right]$$

$$= \left[\frac{b^2}{a^2} - \frac{2c}{a}\right]^3 - 3\left(\frac{c}{a}\right)^2\left[\frac{b^2}{a^2} - \frac{2c}{a}\right]$$

$$= \frac{(b^2 - 2ac)^3 - 3a^2c^2(b^2 - 2ac)}{a^6}$$

$$= \frac{1}{a^6}[b^6 + 8a^3c^3 + 3\,b^2\,(2ac)^2 - 3\,b^4\,(2ac) - 3\,a^2\,b^2\,c^2 + 6\,a^3c^3]$$

$$= \frac{1}{a^6}\;[b^6 - 9\,a^2\,b^2\,c^2 - 6a\,b^4c - 2\,a^3\,c^3\,]$$

(vi) $\alpha^{-3} + \beta^{-3} = \dfrac{1}{\alpha^3} + \dfrac{1}{\beta^3} = \dfrac{\alpha^3 + \alpha^3}{(\alpha\beta)^3} = \dfrac{(\alpha+\beta)^3 \;\; - \alpha\beta\,(\alpha + B)}{(\alpha\beta)^3}$

$$= \frac{(-b/a)^3 - 3\left(\frac{c}{a}\right)(-b/a)}{(c/a)^3}$$

$$= \frac{-\frac{b^3}{a^3} + 3(bc/a^2)}{c^3/a^3} = \frac{3abc - b^3}{c^3}$$

(vii) $\left(\dfrac{\alpha}{\beta} - \dfrac{\beta}{\alpha}\right)^2 = \left(\dfrac{\alpha^2 - \beta^2}{\alpha\beta}\right)^2 = \dfrac{[(\alpha-\beta)(\alpha+\beta)]^2}{\alpha^2\beta^2}$

$$= \frac{(\alpha+\beta)^2[(\alpha+\beta)^2 - 2\alpha\beta]}{(\alpha\beta)^2} = \frac{b^2}{a^2}\left[\frac{(b^2 - 2ac)/a^2}{c^2/a^2}\right]$$

$$= \frac{b^4 - 2\,ab^2c}{a^2c^2}$$

(viii) $\alpha^2 - \alpha^2\,(\alpha - \beta)(\,\alpha + \beta)$

But $(\alpha \; - \beta)^2 = (\alpha + \beta)^2 - 4\,\alpha\beta = \left(\frac{-b}{a}\right)^2 - \frac{4c}{a} = \frac{b^2 - 4ac}{a}$

$$\Rightarrow \alpha - \beta = \frac{\pm\sqrt{b^2 - 4ac}}{a}$$

Thus, $\alpha^2 - \beta^2 = \dfrac{\pm\sqrt{b^2 - 4ac}}{a}.\left(\dfrac{-b}{a}\right) = \dfrac{\pm b\sqrt{b^2 - 4ac}}{b^2}.$

Example 2: Find the quadratic equations with real coefficients and with the following pairs of roots(s) :

(i) 3/5, –4/3 (ii) $1 + \sqrt{3}$, $1 - \sqrt{3}$

(iii) $\dfrac{1}{10 - \sqrt{72}}$, $\dfrac{1}{10 + 6\sqrt{2}}$ (iv) $\dfrac{m-n}{m+n}, -\dfrac{m+n}{m-n}$

(v) 2, $1+\sqrt{3}$ (vi) $2 - 3i$, $2 + 3i$

Solution :

(i) We have, sum of the roots $\frac{3}{5} + \left(\frac{-4}{3}\right) = \frac{9-20}{15} = \frac{-11}{15}$

and product of the roots $= \left(\frac{3}{5}\right)\left(\frac{-4}{3}\right) = -\frac{4}{5}$

Thus, the quadratic equation whose roots are 3/5, –4/3 is

$x^2 - (-11/15)x + (-4/5) = 0$ or $15x^2 + 11x - 12 = 0.$

(ii) We have, sum of the roots $(1+\sqrt{3}) + (1-\sqrt{3}) = 2.$

and product of the roots $(1+\sqrt{3})\ (1-\sqrt{3}) = 1 - 3 = -2.$

Thus, the quadratic equation whose roots are $1+\sqrt{3}$, $1-\sqrt{3}$ = is

$$x^2 - 2x - 2 = 0$$

(iii) Note that $\sqrt{72} = \sqrt{6^2.2} = 6\sqrt{2}.$

$$\text{Sum of the roots} = \frac{1}{10-\sqrt{72}} + \frac{1}{10+6\sqrt{2}}$$

$$= \frac{10+6\sqrt{2}+10-6\sqrt{2}}{100-72} = \frac{20}{28} = \frac{5}{7}$$

$$\text{and product of the roots} = \left(\frac{1}{10-\sqrt{72}}\right)\left(\frac{1}{10+6\sqrt{2}}\right) = \frac{1}{100-72} = \frac{1}{28}$$

Thus, the quadratic equation whose roots are $\frac{1}{10-\sqrt{72}}$ and $\frac{1}{10+6\sqrt{2}}$ is

$x^2 - (5/7)x + 1/28 = 0$ or $28x^2 - 20x + 1 = 0.$

(iv) We have sum of the roots $= \frac{m-n}{m+n} + \left(-\frac{m+n}{m-n}\right)$

$$= \frac{(m-n)(m-n) - (m+n)(m+n)}{m^2-n^2} = \frac{(m-n)^2-(m+n)^2}{(m+n)^2} = -\frac{4mn}{(m+n)^2}$$

$$\text{and product of the roots} = \left(\frac{m-n}{m+n}\right)\left[-\frac{m+n}{m-n}\right] = -1.$$

Thus, the quadratic equation whose roots are $\frac{m-n}{m+n}, -\frac{m+n}{m-n}$ is

$$x^2 - \left(-\frac{4mn}{m^2-n^2}\right)x - 1 = 0 \text{ or } (m^2-n^2)\,x^2 + 4mnx - (m^2-n^2) = 0.$$

(iv) Sum of the roots = $2 + 1 + \sqrt{3} = 3 + \sqrt{3}$

and product of the roots = $2(1+ \sqrt{3})$.

Thus, the required equation is $x^2 - (3 + \sqrt{3})x + 2(1 + \sqrt{3}) = 0$

(vi) Sum of the roots = $(2–3i) + (2–3i) = 4$

and product of the roots = $(2–3i)(2–3i) = 4 + 9 = 13$

Thus, the required quadratic equation is $x^2 - 4x + 13 = 0$.

Example 3

(i) The two roots r_1 and r_1 of the quadratic equation $x^2 + kx + 12 = 0$ are such that $|r_1 - r_2| = 1$. Find k.

(ii) The roots r_1 and r_2 of the quadratic equation $5x^2 - px + 1 = 0$ are such that $|r_1 - r_2| = 1$. Determine p.

(iii) If α and β be the roots of the equation $x^2 - 3ax + a^2 = 0$. Determine a if $\alpha^2 + \beta^2 = 7/4$.

(iv) Determine k is one of the roots of the equation $k(x-1)^2 = 5x - 7$ is double the other.

(v) If p and q are roots of the quadratic equation $x^2 + px + q = 0$. Find p and q.

(vi) If the roots of the equation $ax^2 + bx + c = 0$ are in the ratio $p : q$, show that $ac(p+q)^2 = b^2pq$.

(vii) If one root of the quadratic equation $ax^2+bx + c = 0$ is square of the other. Prove that $b^3 + a^2c + ac^2 = 3abc$.

Solution :

(i) We have $r_1 + r_2 = -k$ and $r_1 r_2 = 12$.

Now, $|r_1 - r_2| = 1 \Rightarrow |r_1 - r_2|^2 = 1$

But $|r_1 - r_2| = 1 \Rightarrow (r_1 + r_2)^2 - 4r_1r_1 = (-k)^2 = -4(12) = k^2 - 48$

Thus, $k^2 - 48 = 1$ or $k^2 = 49 \Rightarrow k = \pm 7$.

(ii) We have $r_1 + r_2 = p/5$ and $r_1 r_1 = 1/5$.

Now $|r_1 - r_2| = 1 \Rightarrow |r_1 - r_2|^2 = 1$

But $|r_1 - r_2|^2 = (r_1 + r_2)^2 - 4r_1r_2 = \left(\frac{p}{5}\right)^2 - 4\left(\frac{1}{5}\right) = \frac{p^2-20}{25}$

Thus, $\frac{p^2 - 20}{25} = 1$ or $p^2 - 20 = 25 \Rightarrow p^2 = 45$ or $p = \pm 3\sqrt{5}$.

(iii) We have α + β = 3a and αβ = a^2

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 9a^2 - 2a^2 = 7a^2$

$\Rightarrow \quad 7/4 = 7a^2 \text{ or } a^2 = 1/4 \text{ or } a = \pm 1/2.$

(iv) We first write the given quadratic equation in standard form, as

$k(x-1)^2 = 5x - 7 \Rightarrow k(x^2 - 2x + 1) = 5x - 7$

or $\quad kx^2 - (2k+5)x + k + 7 = 0.$

Let the roots of this equation be α and 2α. Then

$$\alpha + 2\alpha = \frac{2k+5}{k} \text{ and } \alpha.\ 2\alpha = \frac{k+7}{k} \Rightarrow 3\alpha = \frac{2k+5}{k} \text{ and } 2\alpha^2 = \frac{k+7}{k}.$$

Thus, $\alpha = \frac{2k+5}{3k}$. Putting this value in $2\alpha^2 = \frac{k+7}{k}$, we get

$$2\left(\frac{2k+5}{3k}\right)^2 = \frac{k+7}{k}$$

$$\Rightarrow \quad 2(2k+5)^2 = 9k(k+7) \quad \Rightarrow \quad 2(4k^2 + 20k + 25) = 9k^2 + 63k$$
$$\Rightarrow \quad k^2 + 23k - 50 = 0 \quad \Rightarrow \quad k^2 + 25k - 2k - 50 = 0$$
$$\Rightarrow \quad k(k+25) - 2(k+25) = 0 \quad \Rightarrow \quad (k-2)(k+25) = 0$$
$$\Rightarrow \quad k = 2 \text{ or } k = -25.$$

Thus, the required values of k are 2 and −25.

A Piece of Advise

It is always advisable to use the given condition and reduce the number of unknown. Here instead of beginning with α and β and putting β = 2 α, it is advisable to begin with α and 2α.

When one root is three times other, take the roots as α, 3α; when one is square of the other, take roots as α, α^2.

(v) We have $p + q = -p$ and $pq = q$

Now, $pq = q \Rightarrow q(p-1) = 0 \Rightarrow q = 0 \text{ or } p = 1$

If $q = 0$ then $p = -p$ or $2p = 0$ or $p = 0$.

If $p = 1$ then $1 + q = -1$ or $\Rightarrow q = -2$

Hence, the two solutions are $p = 0$, $q = 0$ and $p = 1$ $q = -2$.

(vi) Let the roots of the equation be $p\alpha$ and $q\alpha$. Then

$p\alpha + q\alpha = -b/a$ and $(p\alpha)(q\alpha) = c/a$

$$\Rightarrow \quad (p+q)\alpha = -\frac{b}{a} \text{ and } pq\,\alpha^2 = c/a$$

$$\Rightarrow \quad \alpha = \frac{-b}{a\,(P+q)} \text{ and } pq\,\alpha^2 = \frac{c}{a}$$

Putting the value of α from the first relation to the second relation, we get

$$pq\left(\frac{-\mathrm{b}}{a(p+q)}\right)^2 = \frac{c}{a} \Rightarrow \frac{pqb^2}{a^2(p+q)} = \frac{c}{a} \Rightarrow b^2pq = ac(p+q)^2.$$

(vii) Let the roots of the equation be α and α^2 . Then

$$\alpha + \alpha^2 = -b/a \ \ and\ \alpha\ \alpha^2 = c/a \Rightarrow \alpha + \alpha^2 = -b/a \ \ and\ \alpha^3 = c/a$$

Cubing both the sides of the relation $\alpha + \alpha^2 = -b/a$ we get

$$(\alpha + \alpha^2)^3 = (-b/a)^3$$

$$\Rightarrow \alpha^3 + (\alpha^2)^3 + 3\,\alpha\,\alpha^2\,(\alpha + \alpha^2) = -b^3/a^3$$

$[\because (x+y)^3 = x^3 + y^3 + 3xy(x+y)]$

$$\Rightarrow \alpha^3 + \alpha^6 + 3\alpha^3\,(\alpha + \alpha^3) = -b^3/a^3$$

Substituting the value of α^3 *and* $\alpha + \alpha^2$, we get

$$\frac{c}{a} + \left(\frac{c}{a}\right)^2 + 3\left(\frac{c}{a}\right)\left(\frac{-b}{a}\right) = \frac{-b^3}{a^3} \Rightarrow \frac{c}{a} + \frac{c^2}{a^2} - \frac{3bc}{a^2} = \frac{-b^3}{a^3}$$

Multiplying both the sides by a^3, *we obtain*

$$a^2c + ac^2 - 3abc = -b^3$$

$$\Rightarrow \quad b^3 + a^2c + a\,c^2 = 3abc$$

Example 4

(i) The sum of the squares of two numbers is 233 and one of the numbers is 3 less than twice the other. Find the numbers.

(ii) A positive number exceeds its positive square root by 12. Find the number.

(iii) The sum of the squares of three consecutive natural numbers is 110. Find the natural numbers.

(iv) Two numbers are such that their sum is 54 and product is 629. Find the numbers.

(v) The length of a rectangular field is greater than its width by 10 metres. If the area of the field is 144 sq. m. find its dimensions.

(vi) The number of straight lines y that can connect x points in a plane is given by :

$y = (x/2)(x-1)$.

How many points does a figures have if only 15 lines can be drawn connecting its vertices ?

(vii) Rachit wishes to start a 100 sq. m rectangular (not a square) vegetable garden. Since he has not only 30 metres barbed wire for fencing, he fences three sides of the rectangle, letting his house wall act the fourth side. How wide is vegetable garden ?

Solution :

(i) Since one of the numbers is 3 less than twice the other, we take the numbers to be x and $2x - 3$. According to the given problem.

$$x^2 + (2x-3)^2 = 233$$

$\Rightarrow$ $x^2 + 4x^2 + 9 - 12x - 233 = 0$ or $5x^2 - 12x - 244 = 0$

$$\therefore \quad x = \frac{12 \pm \sqrt{44 - 4(5)(-224)}}{10} = \frac{12 \pm \sqrt{144 + 4480}}{10}$$

$$x = \frac{12 \pm \sqrt{4264}}{10} = \frac{12 \pm 68}{10} = 8 \text{ or } \frac{-28}{5}$$

When $x = 8$, the two numbers are 8 and $2 \times 8 - 3 = 13$

When $x = \frac{-28}{5}$, the two numbers are $\frac{-28}{5}$ and $2\left(\frac{-28}{5}\right) - 3 = \frac{-71}{5}$.

(ii) Let the positive number be x. According to the given condition

$$x - \sqrt{x} = 12 \Rightarrow x - 12 = \sqrt{x}$$

Squaring both the sides, we $x^2 - 24x + 144 = x$

$\Rightarrow$ $x^2 - 25x + 144 = 0$ $\Rightarrow$ $x^2 - 9x - 16x + 144 = 0$

$\Rightarrow$ $x(x-9) - 16(x-9) = 0$ $\Rightarrow$ $(x-16)(x-9) = 0$

$\Rightarrow$ $x = 16$ or 9.

Putting $x = 16$ in (1), we get $16 - 4 = 12$ which is true.

Putting $x = 9$ (1), we get $9 - 3 = 12$ or $-6 = 12$ which is not true.

Thus, $x = 16$.

Alternative solution

Putting $\sqrt{x} = y \; in$ (1) we get $y^2 - y = 12$ or $y^2 - y - 12 = 0$

$\Rightarrow$ $(y-4)(y+3) = 0$ $\Rightarrow$ $y = 4$ or $y = -3$ $\sqrt{x} = 4$ or $\sqrt{x} = -3$

Since $\sqrt{x} \geq 0$, we reject $\sqrt{x} = -3$

Thus, $\sqrt{x} = 4 \Rightarrow$ $x = 16$.

(iii) Let three consecutive natural numbers be x, $x+1$, and $x+2$. According to the given condition.

$x^2 + (x+1)^2 + (x+2)^2 = 110$

$\Rightarrow \; x^2 + x^2 + 2x + 1 + x^2 + 4x + 4 = 110 \;\; \Rightarrow \; 3x^2 + 6x + 5 = 110$

$\Rightarrow \; 3x^2 + 6x - 105 = 0 \qquad \Rightarrow \; x^2 + 2x - 35 = 0$

$\Rightarrow \; x^2 + 7x - 5x - 35 = 0 \qquad \Rightarrow \; x(x+7) - 5(x+7) = 0$

$\Rightarrow \; (x-5)(x+7) = 0 \qquad \Rightarrow \; x = 5, -7$

Since -7 is not a natural number, so we take $x = 5$

Thus, the required natural numbers are 5, 6,7.

Remark : If you take three numbers as $(x-1)$, x and $(x+1)$, then the calculation is much simpler. Try it as an exercise of your self and see.

(iv) Since the sum of the numbers is 54, we let the numbers be x and $54-x$.

According to the given problem $x(54-x) = 629$

or $\quad 54x - x^2 = 629 \qquad$ or $\quad x^2 - 54x + 629 = 0$

or $\quad x^2 - 37x - 17x + 629 \qquad$ or $\quad x(x-37) - 17(x-37) = 0$

or $\quad (x-37)(x-17) = 0 \qquad$ or $\quad x = 37, 17$

when $x = 37$, the numbers are 37 and $54 - x = 54 - 37 = 17$

when $x = 17$, the numbers are 17 and $54 - 17 = 37$

Hence, the numbers are 17 and 37.

(v) Let the width of the rectangular field be x metres, then length is $x + 10$ metres.

Area of the field = length × breadth = $(x+10)\,x\text{m}^2 = (x^2 + 10x)\text{ m}^2$

Thus $x^2 + 10x = 144 \Rightarrow x^2 + 10x - 144 = 0$

$$\Rightarrow x = \frac{-10 \pm \sqrt{(10)^2 - 4 \times 1 \times (-144)}}{2 \times 1}$$

$$= \frac{-10 \pm \sqrt{100 + 576}}{2 \times 1} = \frac{-10 \pm \sqrt{676}}{2} = \frac{-10 \pm 26}{2} = 8, -18$$

Since x cannot be negative, we take $x = 8$. Thus, the dimension of the field is $8\,m \times 18m$.

(vi) Since, the number of lines is given to be 15. We must have

$$\frac{x}{2}(x-1) = 15 \text{ or } x(x-1) = 30 \text{ or } x^2 - x - 30 = 0$$

$$\Rightarrow x^2 - 6x + 5x - 30 = 0 \Rightarrow (x-6)(x+5) = 0 \Rightarrow x = 6 \text{ or } -5.$$

As x cannot be negative, we have $x = 6$.

(vii) Let the dimension of the rectangular fixed be x metres be y metres. Suppose the house is along the side having length y metres [see Figure 1]

Then $x + x + y = 30$.

and $xy = 100$

$\Rightarrow$ $y = 30 - 2x$ and $xy = 100$

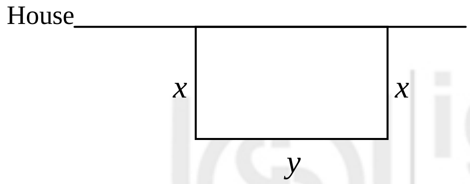

Figure 1

Putting $y = 30 - 2x$ in $xy = 100$, we get $x(30 - 2x) = 100$

$\Rightarrow$ $2x(15 - x) = 100$ or $15x - x^2 + 50 = 0$

$\Rightarrow x^2 - 15x + 50 = 0 \Rightarrow (x-5)(x-10) = 0 \Rightarrow x = 5$ or $x = 10$.

When $x = 5$, $y = 30 - 2(5) = 20$

When $x = 10$, $y = 30 - 2(10) = 10$

As the garden is rectangular and not a square, the dimension of the vegetable garden has to be 5 m $\times$ $20m$.

Example 5 : (i) If α and β are the roots of the equation $2x^2 - 3x - 5 = 0$ from a quadratic equation whose roots are α^2, β^2.

(ii) If α, β are the roots of the equation $2x^2 - 3x + 1 = 0$ form an equation whose roots are α/β and β/α.

(iii) If α, β are the roots of $x^2 - 4x + 5 = 0$ form and equation whose roots are α^2+2, β^2+2.

(iv) If α and β be the roots of the equation $x^2 - px + q = 0$ form and equation whose roots are

$$\frac{1}{a\,\alpha + b} \text{ and } \frac{1}{a\,\beta + b}.$$

Solution (i) Since α and β are roots of $2x^2 - 3x - 5 = 0$

$\alpha + \beta = 3/2$ and $\alpha\beta = -5/2$

We are to form a quadratic equation whose roots are α^2, β^2.

Let S = Sum of roots $= \alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$.

$$= \left(\frac{3}{2}\right)^2 - 2\left(-\frac{5}{2}\right) = \frac{9}{4} + 5 = \frac{29}{4}$$

P = Product of roots = $\alpha^2\ \beta^2 = (\alpha\beta)^2 = \left(-\frac{5}{2}\right)^2 = \frac{25}{4}$.

Putting values of S and P in $x^2 - Sx + P = 0$, the required equation is

$$x^2 - (29/4)x + 25/4 = 0 \text{ or } 4x^2 - 29x + 25 = 0.$$

(ii) Since α and β are roots of $2x^2 - 3x + 1 = 0$, $\alpha + \beta = 3/2$ and $\alpha\beta = ½$.

We are to form an equation whose roots are α/β and β/α.

$$\text{Let S} = \text{Sum of roots} = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2+\beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}.$$

$$= \frac{\left(\frac{3}{2}\right)^2 - 2\left(\frac{1}{2}\right)}{\frac{1}{2}}$$

$$= \left(\frac{5}{4}\right)\left(\frac{2}{1}\right) = \frac{5}{2}$$

$$\text{P} = \text{Product of roots} = \frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$$

Thus, the required quadratic equation is

$x^2 - (5/2)\,x + 1 = 0$ or $2x^2 - 5x + 2 = 0$

(iii) Since α, β are roots of $x^2 - 4x + 5 = 0$ $\alpha + \beta = 4$ and $\alpha\beta = 5$.

The roots of the required equation are $\alpha^2 + 2$ and $\beta^2 + 2$.

Let S = Sum of the roots = $(\alpha^2 + 2) + (\beta^2 + 2) = \alpha^2 + \beta^2 + 4$

$= (\alpha + \beta)^2 - 2\alpha\beta + 4 = (4)^2 - 2 \times 5 + 4 = 10$

and P = Product of the roots = $(\alpha^2 + 2)(\beta^2 + 2) = \alpha^2\beta^2 + 2(\alpha^2 + \beta^2) + 4$

$= \alpha^2\beta^2 + 2[(\alpha + \beta)^2 - 2\alpha\beta)] + 4 = 25 + 2[16-10] + 4 = 41.$

Thus, the required equation is $x^2 - Sx + \text{P} = 0$ or $x^2 - 10x + 41 = 0$

(iv) Since α, β are the roots of $x^2 - px + q = 0$ $\alpha + \beta = p$ and $\alpha, \beta = q$.

The roots of the required equation are $\dfrac{1}{a\alpha + b}$ and $\dfrac{1}{a\beta + b}$.

$$\text{Let S} = \text{Sum of the roots} = \frac{1}{a\alpha + b} + \frac{1}{a\beta + b}$$

$$= \frac{a(\alpha+\beta) + 2b}{a^2\alpha\beta + ab(\alpha+\beta) + b^2}$$

$$= \frac{ap + 2b}{a^2q + abp + b^2}$$

$$\text{P} = \text{Product of the roots } \frac{1}{a\alpha + b} \times \frac{1}{a\beta + b} = \frac{1}{\alpha^2\alpha\beta + ab(\alpha+\beta) + b^2}$$

$$\frac{1}{\alpha^2 q + abp + b^2}$$

The required equation is

x^2 – (Sum of the roots) x + Product of the roots = 0

or $x^2 - \frac{ap+2b}{a^2q + abp + b^2} x + \frac{1}{a^2 + abp + b^2} = 0$

$\Rightarrow \ (a^2q + abp + b^2) \ x^2 - (ap + 2b)x + 1 = 0$

Check Your Progress – 1

1. If α and B are the roots of the equation $ax^2 + bx + c = 0$ $a \neq 0$, find the value of

   (i) $\dfrac{\alpha+\beta}{\frac{1}{1/\alpha} + \frac{1}{1/\beta}}$ (ii) $\alpha^4 + \alpha^4 + \alpha^2\beta^2$

   (iii) $\dfrac{\alpha^3}{\beta} + \dfrac{\beta^3}{\alpha}$ (iv) $\left(\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha}\right)^3$

   (v) $\alpha^3\beta + \alpha\beta^3$ (vi) $\alpha^{-4} + \beta^{-4}$

2. Find the quadratic equation with real coefficients and with the following pairs of root(s) :

   (i) 2/7, –3/7 (ii) $2 + \sqrt{2}, 2 - \sqrt{2}$

   (iii) $\dfrac{1}{5 - \sqrt{6}}, \dfrac{1}{5 + \sqrt{6}}$ (iv) $\dfrac{m}{n}, \dfrac{n}{m},$

   (v) $3, 1 - \sqrt{2}$ (vi) $1 - 3i$, $1 + 3i$

3. If α, β are the roots of the equation $3x^2 - 4x + 1 = 0$, form an equation whose roots are α^2/β *and* β^2/α.
4. If α, β are roots of $x^2 - 2x + 3 = 0$, form an equation whose roots are α + 2, β + 2.

5. If α, β are the roots of $2x^2 - 3x + 5 = 0$ find the equation whose roots are α + 1/β and β + 1/α.

6. If α, β be the roots of $2x^2 - 3x + 1 = 0$ find an equation whose roots are

$$\frac{\alpha}{2\beta+3}, \frac{\beta}{2\alpha+3}.$$

7. If α, β are the roots of $ax^2 + bx + c = 0$ form an equation whose roots form an equation whose roots are α + β and αβ /(α + β).

8. If p, q be the roots of $3x^2 - 4x + 1 = 0$, show that b and c are the roots are

$$x^2 - (p + q - pq)x - pq\,(p + q) = 0.$$

3.4 CUBIC AND BIQADRATIC EQUATIONS

A polynomial equation in x, in which the highest exponent is 3 is called a **cubic** equation, and if the highest exponent is 4, it is called a biquadratic equation. Thus, a cubic equation look as

$$a_0x^3 + a_1x^2 + a_2x + a_3 = 0, \quad a_0 \neq 0 \qquad (1)$$

and *a* biquadratic equation looks as

$$a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4 = 0, \quad a_0 \neq 0 \qquad (2)$$

Relation between Roots and Coefficients

If α, β, γ are the roots of (1), then

$$\alpha + \beta + \gamma = -\, a_1/a_0 \;= -\frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^3}$$

$$\beta\gamma + \gamma\alpha + \alpha\beta = a_1/a_0 = \frac{\text{Coefficient of } x}{\text{Coefficient of } x^3}$$

$$and\ \alpha\beta\gamma = \frac{-a_3}{a_0} = -\frac{\text{Constant term}}{\text{Coefficient of } x^3}$$

If α, β, γ and δ are the roots of (2), then

$$\alpha + \beta + \gamma + \delta = \frac{-a_1}{a_0} = -\frac{\text{Coefficient of } x^3}{\text{Coefficient of } x^4}$$

$$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = -\frac{a_2}{a_0} = \frac{\text{Coefficient of } x^2}{\text{Coefficient of } x^4}$$

$$\alpha\beta(\gamma + \delta) + \gamma\delta(\alpha + \beta) = -\frac{a_3}{a_0} = -\frac{\text{Coefficient of } x}{\text{Coefficient of } x^4}$$

$$\alpha\beta\gamma\delta = \frac{a_4}{a_0} = \frac{\text{Constant term}}{\text{Coefficient of } x^4}$$

Remarks : It will be difficult to solve a cubic equation and biquadrate equation just by knowing the relation between roots and coefficients. However, if we know one more relation between the roots, it becomes easier for us to solve the equation.

Solved Examples

Example 6 : Solve the equation

$$2x^3 - 15\,x^4 + 37x - 30 = 0 \qquad (1)$$

If the roots of the equation are in A.P.

Solution : Recall three numbers in A.P. can be taken as $\alpha - \beta, \alpha, \alpha + \beta$.

If $\alpha - \beta, \alpha, \alpha + \beta$ are roots of (1), then $(\alpha - \beta) + \alpha + (\alpha + \beta) = 15/2 \Rightarrow 3\alpha = 15/2$

$\Rightarrow \alpha = 5/2$

Next,

$\alpha(\alpha - \beta) + \alpha(\alpha + \beta)\ \ (\alpha - \beta)(\alpha + \beta) = 37/2$

$\Rightarrow\ \alpha^2 - \alpha\beta + \alpha^2 + \alpha\beta + \ \alpha^2 - \ \beta^2 = 37/2$

$\Rightarrow 3\alpha^2 - \beta^2 = 37/2$

$$\Rightarrow \beta^2\ =\ 3\alpha^2 - \frac{37}{2} = 3\ \times \frac{25}{4} - \frac{37}{2} = \frac{1}{4}$$

$$\Rightarrow \beta\ =\ \pm\frac{1}{2}$$

When $\beta = ½$, the roots are

$$\frac{5}{2} - \ \frac{1}{2}, \frac{5}{2}, \frac{5}{2} + \ \frac{1}{2}\ ,\text{or } 2, \frac{5}{2}, 3$$

When $\beta = -\frac{1}{2}$, the roots are 3, 5/2 2.

It is easily to check that these are roots of (1).

Example 7 : Solve the equation

$$6x^3 - \ 11x^2 - \ 3x + 2 = 0 \qquad (1)$$

Given that the roots are in H.P.

Solution : Let the roots of the equation be

$$\frac{1}{\alpha - \beta}, \frac{1}{\alpha}, \frac{1}{\alpha + \beta}$$

We have

$$\frac{1}{\alpha-\beta} + \frac{1}{\alpha} + \frac{1}{\alpha+\beta} = \frac{11}{6} \qquad (2)$$

$$\frac{1}{\alpha-\beta}\cdot\frac{1}{\alpha} + \frac{1}{\alpha}\cdot\frac{1}{\alpha+\beta} + \frac{1}{\alpha-\beta}\cdot\frac{1}{\alpha+\beta} = \frac{-3}{6} = \frac{-1}{2} \qquad (3)$$

$$\text{and } \frac{1}{\alpha-\beta}\,\frac{1}{\alpha}\,\,\frac{1}{\alpha+\beta} = \frac{-2}{6} = \frac{-1}{3} \qquad (4)$$

From (3), we obtain

$$\frac{(\alpha+\beta)+\alpha+(\alpha-\beta)}{(\alpha-\beta)\alpha(\alpha+\beta)} = \frac{-1}{2}$$

$$\Rightarrow 3\alpha = \frac{3}{2} \Rightarrow \alpha = \frac{1}{2}$$

Putting this in (4), we obtain

$$\left(\frac{1}{2}-\beta\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}+\beta\right) = -3$$

$$\Rightarrow \frac{1}{4} - \beta^2 = -6$$

$$\Rightarrow \beta^2 = \frac{1}{4} + 6 = \frac{25}{4}$$

$$\Rightarrow \beta^2 = \pm\frac{5}{2}.$$

If $\beta^2 = \frac{5}{2}$, roots become $-\frac{1}{2}, 2$ and $\frac{1}{3}$

If β = –5/2, we get these roots in the reverse order.

Example 8: Solve the equation

$$8x^3 - 14x^2 + 7x - 1 = 0 \qquad (1)$$

The roots being in G.P.

Solution : As the roots are in G.P. we may take them as

$$\frac{\alpha}{r}, \alpha, \alpha r$$

Now,

$$\text{Now, } \frac{\alpha}{r} + \alpha + \alpha r = \frac{14}{8} = \frac{7}{4}, \qquad (2)$$

$$\left(\frac{\alpha}{r}\right)(\alpha) + \alpha(\alpha r) + \left(\frac{\alpha}{r}\right)(\alpha r) = \frac{7}{8} \qquad (3)$$

$$\left(\frac{\alpha}{r}\right)(\alpha)(\alpha r) = \frac{1}{8} \qquad (4)$$

From (4), we get $\alpha^3 = \frac{1}{8} \Rightarrow \alpha = \frac{1}{2}$

Putting this in (2), we get

$$\frac{1}{2}\left(\frac{1}{r} + r\right) = \frac{7}{4} - \frac{1}{2} = \frac{5}{4}$$

$\Rightarrow \quad r + \frac{1}{r} = \frac{5}{2} \qquad \Rightarrow \quad \frac{r^2+1}{r} = \frac{5}{2}$

$\Rightarrow \quad 2r^2 - 5r + 2 = 0 \qquad \Rightarrow \quad (2r-1)(r-2) = 0$

$\Rightarrow \quad r = 1/2, r = 2$

When $r = 2$, we get the roots as

$\frac{1}{4}, \frac{1}{2}, 1$

When $r = 1/2$, we get these roots in the reverse order.

It is easy to verify that these roots satisfy the equation (1)

Example 9: Solve the equation

$$x^2 - 13x^2 + 15x + 189 = 0 \qquad (1)$$

being given that one root exceeds the other by 2.

Solution : As one root exceeds the other by 2, we may take the root as α, $\alpha+2$ and β.

Now.

$\alpha + (\alpha + 2) + \beta = 13 \qquad (2)$

$\alpha(\alpha + 2) + \alpha\beta + (\alpha + 2)\beta = 15 \qquad (3)$

and $\alpha(\alpha + 2)\beta = -189 \qquad (4)$

From (2), $\beta = 11 - 2\alpha$.

Putting this in (3), we get

$\alpha(\alpha + 2) + (2\alpha + 2)(11 - 2\alpha) = 15$

$\Rightarrow \quad \alpha^2 + 2\alpha + 22\alpha + 22 - 4\alpha^2 - 4\alpha = 15$

$\Rightarrow \quad 3\alpha^2 - 20\alpha - 7 = 0$

$\Rightarrow \quad (3\alpha + 1)(\alpha - 7) = 0$

$\Rightarrow \quad \alpha = -1/3, 7$

when $\alpha = -\frac{1}{3}, \beta = 11 + \frac{2}{3} = \frac{35}{3}.$

But $\alpha = \frac{1}{3}$, $\beta = \frac{35}{3}$ does not satisfy equation (4)

When $\alpha = 7$, $\beta = -3$

These values satisfy equation (4)

Thus, roots of (1) are 7,9 and –3

Example 10 : Solve the equation

$$x^4 - 2x^3 + 4x^2 + 6x - 21 = 0 \qquad (1)$$

being given that it has two roots equal in magnitude but opposite sign.

Solution : Let the roots of (1) be

$\alpha, -\alpha, \beta$ and γ.

We have

$\alpha + (-\alpha) + \beta + \gamma = 2 \qquad (2)$

$[\alpha + (-\alpha)(\beta + \gamma) + \alpha(-\alpha) + \beta\gamma = 4 \qquad (3)$

$(\alpha + (-\alpha))\beta\gamma + \alpha(-\alpha)(\beta + \gamma) = -6 \qquad (4)$

$\alpha(-\alpha)\beta\gamma = -21 \qquad (5)$

(2) gives $\beta + \gamma = 2$

Putting this in (4), we get $\alpha^2 = 3$ or $\alpha = \pm\sqrt{3}$.

From (3), we get

$$0(\beta + \gamma) + (-3) + \beta\gamma = 4$$

$$\Rightarrow \beta\gamma = 7$$

The quadratic equation whose roots are β and γ is

$x^2 - (\beta + \gamma)x + \beta\gamma = 0$

or $x^2 - 2x + 7 = 0$

$(x-1)^2 + 6 = 0$

$\Rightarrow\ x = 1 \pm \sqrt{6}\,\mathrm{i}$

Thus, roots of (1) are

$\sqrt{3}, -\sqrt{3}, 1+\sqrt{6}i,\ 1-\sqrt{6}i$

Example 11: Solve the equation

$$3x^4 - 25\,x^3 + 50\,x^2 - 50\,x^2 + 12 = 0 \qquad (1)$$

The product of two of roots being 2.

Solution : Let roots of (1) be

$\alpha, \beta, \gamma, \delta,$ where $\gamma\,\delta = 2.$

We have

$$\alpha + \beta + \gamma + \delta = \frac{25}{3} \qquad (2)$$

$$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\,\delta = \frac{50}{3} \qquad (3)$$

$$(\alpha + \beta)\,\gamma\delta + \alpha\beta + (\gamma + \delta) = \frac{50}{3} \qquad (4)$$

and $\alpha\beta\gamma\delta = -12/3 = 4$ (5)

As $\gamma\,\delta = 2$, from (5) we get $\alpha\beta = +2$.

Putting $\alpha\beta = 2$, $\gamma\delta = 2$ in (3), we get

$$(\alpha + \beta)(\gamma + \delta) = \frac{50}{3} - 4 = \frac{38}{3}$$

Equation whose rots are $\alpha + \beta$ and $r + \delta$ is

$$x^2 - (\alpha+\beta+\gamma+\delta)x + (\alpha+\beta)(\gamma+\delta) = 0$$

$$\Rightarrow\ x^2 - \frac{25}{3}x + \frac{38}{3} = 0$$

$$\Rightarrow\ 3x^2 - 25x + 38 = 0$$

$$\Rightarrow\ 3x^2 - 6x - 19x + 38 = 0$$

$$\Rightarrow\ 3x(x-2) - 19(x-2) = 0$$

$\Rightarrow \quad (3x - 19)(x - 2) = 0 \Rightarrow x = \dfrac{19}{3}, x = 2$

Let $\alpha + \beta = 19/3$ and $r + \delta = 2$

Equation whose roots are α and β is

$\Rightarrow x^2 - (\alpha + \beta)x + \alpha\beta = 0$

or $x^2 - \dfrac{19}{3}x + 2 = 0$

$\Rightarrow \quad 3x^2 - 19x + 6 = 0$

$\Rightarrow (3x - 1)(x - 6) = 0$

$\Rightarrow \quad x = 1/3, 6$

Equation whose roots are γ and δ is

$$x^2 - (r + \delta)x + r\delta = 0$$

$\Rightarrow \quad x^2 - 2x + 2 = 0$

$\Rightarrow \quad (x - 1)^2 + 1 = 0 \quad \Rightarrow (x - 1)^2 = i^2$

$\Rightarrow \quad x - 1 = \pm i, \qquad x = 1 \pm i$

Thus, roots of (1) are 1/3, 6, $1 + i$, $1 - i$.

Example 12 : The Product of two of the roots of the equation

$x^4 - 5x^3 + 10x^3 - 10x + 4 = 0$ is equal to the product of the other two. (1)

Solution : Let roots of (1) be

$\alpha, \beta, \gamma, \delta$

where $\quad \alpha\beta = \gamma\,\delta$

We have

$\alpha + \beta + \gamma + \delta = 5$ (2)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\,\delta = 10$ (3)

$\alpha\beta(\gamma + \delta) + \gamma\,\delta\,(\alpha + \beta) = 10$ (4)

$\alpha\beta\gamma\delta = 4$ (5)

Since $\alpha\beta = \gamma\delta$ (5) gives $\alpha\beta = \gamma\delta = \pm 2$.

But $\alpha\beta = \gamma\delta = -2$, gives

$-2(\gamma + \delta + \alpha + \beta) = 10$ [from (4)]

$\alpha + \beta + \gamma + \delta = -5$

This contradicts (2)

Thus, $\alpha\beta = \gamma\delta = 2$.

Putting these values in (3), we obtain

$(\alpha + \beta)(\gamma + \delta) = 6$.

A quadratic equation whose roots are $\alpha+\beta$, $\gamma + \delta$ is

$$x^2 - (\alpha + \beta + \gamma + \delta)x + (\alpha + \beta)(\gamma + \delta) = 0$$

or $x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$

Let $\alpha + \beta = 2$ and $\gamma + \delta = 3$.

A quadratic equation whose roots are α, β is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

$$\Rightarrow x^2 - 2x + 2 = 0 \Rightarrow (x-1)^2 = -1 = i^2$$

$$\Rightarrow x = 1 \pm i$$

A quadratic equation whose roots are γ and δ is

$$x^2 - (\gamma + \delta)x + \gamma\delta = 0$$

or $x^2 - 3x + 2 = 0 \Rightarrow x = 1, 2$

Hence, roots of (1) are

$1, 2, 1+i, 1-i$

Example 13: Solve the equation

$x^4 - 8x^3 + 21x^2 - 20x + 5 = 0$, the sum of two of the roots being equal to the sum of the other two.

Solution : Let roots of the equation be α, β, γ and δ where $\alpha + \beta = \gamma + \delta$.

We have

$\alpha + \beta + \gamma + \delta = 8$ (2)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = 21$ (3)

$\alpha\beta(\gamma + \delta) + \gamma\,\delta(\alpha + \beta) = 20$ (4)

$\alpha\,\beta\,\gamma\;\;\delta = 5$ (5)

As $\alpha+\beta = \gamma + \delta$, from (2), we get

$2\,(\alpha + \beta) = 8 \Rightarrow \alpha + \beta = 4$

Thus, $\alpha+\beta = \gamma + \delta = 4$

Putting this in (3), we get

$\alpha\beta + \gamma\,\delta = 5$

A quadratic equation whose roots are $\alpha\beta$ and $\gamma\,\delta$ is

$x^2 - (\alpha\beta + \gamma\,\delta)x + (\alpha\beta)\,(\gamma\,\delta) = 0$

$\Rightarrow\; x^2 - 5x + 5 = 0$

$$\Rightarrow \quad x = \frac{5 \pm \sqrt{25 - 20}}{2} = \frac{1}{2}\,(5 \pm \sqrt{5}\,)$$

Let $\alpha\beta = \frac{1}{2}\,(5 + \sqrt{5}\,)$ and $\gamma\,\delta = \frac{1}{2}(5 - \sqrt{5}\,)$

A quadratic equation whose roots are α, β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$or \qquad x^2 - 4x + \frac{1}{2}(5 + \sqrt{5}\,) = 0$

$$\Rightarrow \; x = \frac{4 \pm \sqrt{16 - 10 - 2\sqrt{5}}}{2} = \frac{4 \pm \sqrt{6 - 2\sqrt{5}}}{2}$$

$$= \frac{4 \pm \sqrt{(\sqrt{5} - 1)^2}}{2} = \frac{4 \pm (\sqrt{5} - 1)}{2}$$

$$= \frac{1}{2}\,(3 + \sqrt{5}), \frac{1}{2}(5 - \sqrt{5}),$$

A quadratic equation whose roots are γ, δ is

$x^2 - (\gamma + \delta)x + \gamma\delta = 0$

or $x^2 - 4x + \frac{1}{2}(5 - \sqrt{5}) = 0$

$$\Rightarrow \ x = \frac{4 \pm \sqrt{16 - 10 + 2\sqrt{5}}}{2}$$

$$= \frac{4 \pm \sqrt{(\sqrt{5}+1)^2}}{2} = \frac{4 \pm (\sqrt{5}+1)}{2}$$

$$= \frac{3 - \sqrt{5}}{2}, \frac{5 + \sqrt{5}}{2}$$

Thus, roots of (1) are

$\frac{1}{2}(3 \pm \sqrt{5}), \ \frac{1}{2}(5 \pm \sqrt{5})$

Example 14 : If roots of $ax^3 + bx^2 + cx + d = 0$ are in A.P. show that

$$2b^3 - 9abc + 27\,a^2d = 0 \qquad (1)$$

Solution : Let roots of

(1) be $\alpha - \beta$, α $\alpha + \beta$

we have

$$(\alpha - \beta) + \alpha + (\alpha + \beta) = -\frac{b}{a}$$

$$\Rightarrow 3\alpha = -\frac{b}{a} \Rightarrow \alpha = -\frac{b}{3a}$$

As α is a roots of (1), we get

$$a\left(-\frac{b}{3a}\right)^3 + b\left(-\frac{b}{3a}\right)^2 + c\left(-\frac{b}{3a}\right) + d = 0$$

$$\Rightarrow \frac{-b^3}{27a^2} + \frac{b^3}{9a^2} - \frac{bc}{3a} + \mathrm{d} = 0$$

$$\Rightarrow -b^3 + 3b^3 - 9abc + 27a^2d = 0$$

$$\Rightarrow 2b^3 - 9abc + 27a^2d = 0$$

Example 15 : If α, β, γ be the roots of the equation $x^3 - px^2 + qx - \gamma = 0$ (1)

Find the values of

(i) $\alpha^2 + \beta^2 + \gamma^2$ (ii) $\beta^2 r^2 + \alpha^2 + \alpha^2\beta^2$

(iii) $\alpha^3 + \beta^3 + \gamma^3$ (iv) $\sum \alpha^2\beta$

Solution : We have

$\alpha + \beta + \gamma = p$

$\beta\gamma + \gamma\alpha + \alpha\beta = q,$

and $\alpha\beta\gamma = \gamma$

Now,

(i) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2\,(\beta\gamma + \gamma\alpha + \alpha\beta)$

$= p^2 - 2q$

(ii) $\beta^2\gamma^2 + \alpha^2\gamma^2 + \alpha^2\beta^2$

$= (\beta\gamma + \gamma\alpha + \alpha\beta)^2 - 2[(\beta\gamma)(\gamma\alpha) + (\gamma\alpha)(\alpha\beta) + (\beta\gamma)(\alpha\beta)]$

$= (\beta\gamma + \gamma\alpha + \alpha\beta)^2 - 2\alpha\beta\gamma\,(\alpha + \beta + \gamma)$

$= q^2 - 2\gamma p$

(iii) $\alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma = (\alpha + \beta + \gamma)\ (\alpha^2 + \beta^2 + \gamma^2 - \beta r - \gamma\alpha - \alpha\beta)$

$= (\alpha + \beta + \gamma)\ [(\alpha + \beta + \gamma)^2 - 3(\beta\gamma + \gamma\alpha + \alpha\beta)]$

$\Rightarrow\ \alpha^3 + \beta^3 + \gamma^3 - 3\gamma = p\,[p^2 - 3q]$

$\Rightarrow\ \alpha^3 + \beta^3 + \gamma^3 = p^3 - 3pq + 3\gamma$

(iv) $\sum \alpha^3\beta = \alpha^3(\beta + \gamma) + \alpha^3(\gamma + \alpha) + \alpha^3(\alpha + \beta)$

$= (\alpha + \beta + \gamma)\,(\beta\gamma + \gamma\alpha + \alpha\beta) - 3\alpha\beta\gamma$

$= pq - 3\gamma$

Example 16 : α, β, γ are the roots of the equation $x^3 + px + q = 0,$ (1)

Then, show that

$$\frac{1}{7}\,(\alpha^7 + \beta^7 + \gamma^7) = \frac{1}{7}\,(\alpha^2 + \beta^2 + \gamma^2)\,\frac{1}{5}\,(\alpha^5 + \beta^5 + \gamma^5)$$

Solution : We have

$\alpha + \beta + \gamma = 0$

$\beta r + \alpha\gamma + \alpha\beta = p$

and $\alpha\beta\gamma = q$

Now, $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\beta\gamma + \gamma\alpha + \alpha\beta)$

$$= 0 - 2p = -2p \qquad (2)$$

Next

$$\alpha^3 + \beta^3 + \gamma^3 = -p(\alpha + \beta + \gamma) - 3q = -3q \qquad (3)$$

[using (1)]

Multiplying (1) by x^2, we get

$$x^5 + px^3 + qx^3 = 0$$

As α, β, γ satisfy this equation,

$$\alpha^5 + \beta^5 + \gamma^5 = -p(\alpha^3 + \beta^3 + \gamma^3) - q(\alpha^2 + \beta^2 + \gamma^2)$$

$$= -p(-3q) - q(-2p) = 5pq \qquad (4)$$

Multiplying (1) by x^4, we get

$$x^7 + px^5 + qx^4 = 0$$

As α, β, r satisfy this equation, we get

$$\alpha^7 + \beta^7 + \gamma^7 = -p(\alpha^5 + \beta^5 + \gamma^5) - q(\alpha^4 + \beta^4 + \gamma^4) \qquad (5)$$

But $\alpha^5 + \beta^5 + \gamma^5 = 5pq$ (6)

Multiplying (1) by x we get

$$x^4 + px^2 + +qx = 0.$$

As α, β, γ satisfy this equation, we obtain

$$\alpha^4 + \beta^4 + \gamma^4 = -p(\alpha^2 + \beta^2 + \gamma^2) - q(\alpha + \beta + \gamma)$$

$$= -p(-2p) - 9(0) = 2p^2 \qquad (7)$$

From (5), (6) and (7) we get

$$\alpha^7 + \beta^7 + \gamma^7 = -p(5pq) - q(2p^2) = -7p^2q$$

$$\Rightarrow \frac{1}{7}(\alpha^7 + \beta^7 + \gamma^7) = -p^2q \qquad (8)$$

Also, $\frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)\ \frac{1}{5}(\alpha^5 + \beta^5 + \gamma^5)$

$$= \frac{1}{2}(-2p).\frac{1}{5}5(pq) = -p^2q \qquad (9)$$

From (8) and (9), we get

$$\frac{1}{7}(\alpha^7 + \beta^7 + \gamma^7) = \frac{1}{2}(\alpha^2 + \beta^2 + \gamma^2)\,\frac{1}{5}(\alpha^5 + \beta^5 + \gamma^5)$$

Example 17 : If $\alpha, \beta, \gamma, \delta$ are the roots of the equation.

$$x^4 + px^3 + qx^2 + rx + s = 0 \qquad (1)$$

Find the value of

(i) $\sum \alpha^2$ (ii) $\sum x^2$

Solution : As α, β, r, δ are the roots of (1)

$\sum \alpha = -p$

$\sum \alpha\beta = q$

$\sum \alpha\beta\gamma = -r$

and $\alpha\beta\gamma\delta = s$

Now,

$\sum \alpha^2\beta = \text{q} = (\sum \alpha)(\sum \alpha\beta) - \sum \alpha\beta\gamma$

$= (-p)(q) - (-\gamma) = -pq + \gamma$

$Next\ \sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

$= (-p)^2 - 2q = p^2 - 2q$

Check Your Progress – 2

1. Solve the equation $32x^2 - 48x^2 + 22x - 3 = 0$, given that the roots are in A.P.
2. Solve the equation $27x^3 + 42x^2 - 28x - 8 = 0$, given that the roots are in G.P.
3. Solve the equation $3x^3 + 11x^2 + 12x + 4 = 0$, given that the roots are in H.P.
4. Solve the equation $32x^3 - 48x^2 + 22x - 3 = 0$, given that sum of two roots is 1.
5. Solve the equation $x^3 - 9x^2 + 23x - 15 = 0$, two of the roots being in the ration 3:5.
6. Solve the equation $x^3 - 13x^2 + 15x + 189 = 0$, given that difference between two of its roots is 2.
7. Solve the equation $27x^4 - 195x^3 + 494x^2 - 520x + 192 = 9$, given that the roots are in G.P.
8. Solve the equation $8x^2 - 2x^2 - 27x^2 + 6x + 9 = 0$, given that two roots are equal in magnitude but opposite in signs.
9. Solve the equation $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$, given that sum of two of the roots being equal to the sum of the other two.
10. Solve the equation $2x^4 - 15x^3 + 35x^2 - 30x + 8 = 0$, given that product of two of its roots equals the product of the other two.

11. Solve the equation $x^4 - 10x^3 + 42x^2 - 82x + 65 = 0$, given that product of two of its roots is 13.
12. If roots of $ax^3 + bx^3 + cx + d = 0$ are in G. P. Show that $ac^3 = b^3 d$.
13. If the roots of $x^3 - px^2 + qx - r = 0$ $(r \neq 0)$are in H.P. show that
    $27r^3 - 9pq\,\gamma + 2q^3 = 0$
14. If α, β, γ are the roots of $x^3 - px^2 + 9x - r = 0$ $(r \neq 0)$
    Find the value of
    (i) $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$ (ii) $\sum \frac{\alpha}{\beta}$ (iii) $\sum \frac{1}{\alpha^2}$
15. If $\alpha, \beta, \gamma, \delta$ are the roots of the equation $x^4 - px^3 + qx^2 - rx + s = 0$, $s \neq 0$
    Find the value of (i) $\sum \alpha^2$ (ii) $\sum \frac{1}{\alpha}$

3.5 ANSWERS TO CHECK YOUR PROGRESS

Check Your Progress – 1

1 We have α + β = $-b/a$, αβ = c/a

(i) $$\frac{\alpha + \beta}{\frac{1}{\alpha} + \frac{1}{\beta}} = \frac{\alpha + \beta}{(\beta + \alpha)/\alpha\beta} = \alpha\beta = c/a$$

(ii) $$\alpha^4 + \beta^4 + \alpha^2\beta^2 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$$

$$= [(\alpha + \beta)^2 - 2\alpha\beta]^2 - \alpha^2\beta^2$$

$$= \left[\left(\frac{-b}{a}\right)^2 - 2\,\frac{c}{a}\right]^2 - \left(\frac{c}{a}\right)^2$$

$$= \left(\frac{b^2 - 2ac}{a^2}\right)^2 - \left(\frac{c}{a^2}\right)^2$$

$$= \frac{(b^2 - 2ac)^2 - a^2c^2}{a^4}$$

(iii) $$\frac{\alpha^3}{\beta} + \frac{\beta^3}{\alpha} = \frac{\alpha^4 + \alpha^4}{\alpha\beta} = \frac{(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2}{\alpha\beta}$$

$$= \frac{[(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2] - 2\alpha^2\beta^2}{\alpha\beta}$$

$$= \frac{[(-b/a)^2 - 2c/a]^2 - 2(c/a)^2}{c/a}$$

$$= \frac{(b^2 - 2ac)^2/a^4 - 2c^2/a^2}{c/a}$$

$$= \frac{(b^2 - 2ac)^2 - 2a^2c^2}{a^3c}$$

(iv) We have $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$

$$= \frac{(-b/a)^2 - 2c/a}{c/a}$$

$$= \frac{b^2 - 2ac}{ac}$$

Thus, $\left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)^3 = \frac{(b^2 - 2ac)^3}{a^3c^3}$

(v) $\alpha^3\beta + \alpha\beta^3 = \alpha\beta(\alpha^2 + \beta^2)$

$$= \alpha\beta[(\alpha+\beta)^2 - 2\alpha\beta]$$

$$= \frac{c}{a}\left[\left(\frac{-b}{a}\right)^2 - 2\,\frac{c}{a}\right]$$

$$= \frac{c(b^2 - 2ac)}{a^3}$$

(vi) $\alpha^{-4} + \beta^{-4} = \frac{1}{\alpha^4} + \frac{1}{\beta^4} = \frac{\alpha^4 + \beta^4}{\alpha^4\beta^4}$

$$= \frac{(\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2}{\alpha^4\beta^4}$$

$$= \frac{[\alpha^2 + \beta^2 - 2\alpha\beta]^2 - 2\alpha^2\beta^2}{(\alpha\beta)^4}$$

$$= \frac{[(-\frac{b}{a})^2 - 2c/a]^2 - 2(\frac{c}{a})^2}{(c/a)^4}$$

$$= \frac{(b^2 - 2ac)^2/(a)^4 - 2c^2/a^2}{c^4/a^4}$$

$$= \frac{(b^2 - 2ac)^2 2c^2a^2(a)^4 - 2c^2/a^2}{c^4}$$

2. (i) Sum of the roots $= \frac{2}{7} + \frac{(-3)}{7} = -\frac{1}{7}$

and product of roots $\left(\frac{2}{7}\right)\left(-\frac{3}{7}\right) = -\frac{6}{49}$

Required equation is $x^2 - \left(-\frac{1}{7}\right)x - \frac{6}{49} = 0$

$\Rightarrow 49x^2 + 7x - 6 = 0$

(ii) Sum of the roots = $(2 + \sqrt{2}) + (2 - \sqrt{2}) = 4$

and product of roots = $(2 + \sqrt{2})(2 - \sqrt{2}) = 4 - 2 = 2$

$\therefore$ required equation is

$x^2 - 4x + 2 = 0$

(iii) Sum of the roots $= \frac{1}{5-\sqrt{6}} + \frac{1}{5+\sqrt{6}}$

$$= \frac{(5+\sqrt{6}) + (5-\sqrt{6})}{(5-\sqrt{6})(5+\sqrt{6})} = \frac{10}{25-6} = \frac{10}{19}$$

and product of the roots $= \frac{1}{5-\sqrt{6}} . \frac{1}{5+\sqrt{6}} = \frac{1}{19}$

$\therefore$ required equation is $x^2 - \frac{10}{19}x + \frac{1}{19} = 0$

or $19x^2 - 10x + 1 = 0$

(iv) Sum of the roots $\frac{m}{n} + \left(-\frac{n}{m}\right) = \frac{m^2 - n^2}{mn}$

and proudct of the roots $= \left(\frac{m}{n}\right)\left(-\frac{n}{m}\right) = -1$

$\therefore$ required equation is $x^2 - \frac{m^2 - n^2}{mn}x - 1 = 0$

$\Rightarrow mnx^2 - (m^2 - n^2)x - 1 = 0$

(v) Sum of the roots = $3 + (1 - \sqrt{2}) = 4 - \sqrt{2}$,

and product of the roots = $3(1 - \sqrt{2})$

$\therefore$ required equation is

$x^2 - (4 - \sqrt{2})x + 3(1 - \sqrt{2}) = 0$

(vi) Sum of the roots $= (1-3i) + (1+3i) = 2$

Product of the roots $= (1-3i)(1+3i) = 1 + 9 = 10$

$\therefore$ required equation is

$$x^2 - 2x + 10 = 0$$

3. We have

$\alpha + \beta = 4/3, \quad \alpha\beta = 1/3$

$$\text{Now,} \quad \frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

$$= \frac{(4/3)^3 - 3(1/3)(4/3)}{1/3}$$

$$= 3\left[\frac{64}{27} - \frac{4}{3}\right] = \frac{64}{9} - 4 = \frac{28}{9}$$

$$\text{and } \left(\frac{\alpha^2}{\beta}\right)\left(\frac{\beta^2}{\alpha}\right) = \alpha\beta = 1/3$$

$\therefore$ required equation is

$$x^2 - \frac{28}{9}x + \frac{1}{3} = 0$$

$$\text{or} \quad 9x^2 - 28x + 3 = 0$$

4. $\alpha + \beta = 2$, $\alpha\beta = 3$.

Now, $(\alpha + 2) + (\beta + 2) = \alpha + \beta + 4 = 6$

and $(\alpha + 2)(\beta + 2) = \alpha\beta + 2(\alpha + \beta) + 4$

$= 3 + 2(2) + 4 = 11$

Thus, required equation is

$$x^2 - 6x + 11 = 0.$$

5. $\alpha + \beta = 3/2$ and $\alpha\beta = 5/2$

$$\text{Now,} \left(\alpha + \frac{1}{\beta}\right) + \left(\beta + \frac{1}{\alpha}\right) = (\alpha + \beta) + \left(\frac{1}{\beta} + \frac{1}{\alpha}\right)$$

$$= (\alpha + \beta) + \frac{(\alpha + \beta)}{\alpha\beta}$$

$$= \frac{3}{2} + \frac{\frac{3}{2}}{\frac{5}{2}} = \frac{3}{2} + \frac{3}{5} = \frac{21}{10}$$

and $\left(\alpha + \frac{1}{\beta}\right)\left(\beta + \frac{1}{\alpha}\right) = \alpha\beta + \frac{1}{\alpha\beta} + 2$

$$= \frac{5}{2} + \frac{2}{5} + 2$$

$$= \frac{49}{10}$$

$\therefore$ required equation is

$$x^2 - \frac{21}{10}x + \frac{49}{10} = 0.$$

Or $10x^2 - 21x + 49 = 0$

6. $\alpha + \beta = 3/2$, $\alpha\beta = ½$

Now, $$\frac{\alpha}{2\beta+3} + \frac{\beta}{2\alpha+3} = \frac{2\alpha^2 + 3\alpha + 2\beta^2 + 3\beta}{(2\beta+3)(2\alpha+3)}$$

$$= \frac{2(\alpha+\beta)^2 + 3(\alpha+\beta)}{4\,\alpha\beta + 6(\alpha+\beta) + 9}$$

$$= \frac{2[(\alpha^2 + \beta^2) - 2\alpha\beta] + 3(\alpha+\beta)}{4\,\alpha\beta + 6(\alpha+\beta) + 9}$$

$$= \frac{2[9/4 - 2\left(\frac{1}{2}\right)] + 3(3/2)}{4\,(1/2) + 6(3/2) + 9}$$

$$= \frac{7}{2+9+9} = \frac{7}{20}$$

and $$\left(\frac{\alpha}{2\beta+3}\right)\left(\frac{\beta}{2\alpha+3}\right) = \frac{\alpha\beta}{4\alpha\beta + 6(\alpha+\beta) + 9}$$

$$= \frac{1/2}{4(1/2) + 6\,(3/2) + 9}$$

$$= \frac{1/2}{20} = \frac{1}{40}$$

Thus, required equation is

$$x^2 - \frac{7}{20}x + \frac{1}{40} = 0$$

$40x^2 - 14x + 1 = 0$

7. $\alpha + \beta = -b/a, \quad \alpha\beta = c/a$

$$\text{Now, } \alpha + \beta + \frac{\alpha\beta}{\alpha+\beta} = \frac{(\alpha+\beta)^2 + \alpha\beta}{\alpha+\beta}$$

$$= \frac{(-b/a)^2 + ac}{-b/a}$$

$$= \frac{b^2 + ac}{-ab}$$

$$\text{and } (\alpha + \beta)\left(\frac{\alpha\beta}{\alpha+\beta}\right) = \alpha\beta = \frac{c}{a}$$

Thus, required equation is

$$x^2 + \frac{b^2 + ac}{ab}x + \frac{c}{a} = 0$$

$$\Rightarrow ab\,x^2 + (b^2 + ac)x + bc = 0$$

8. We have $p + q = -b$ and $pq = c$.

Now, $b + c = -(p+q) + pq$

and $bc = -(p+q)pq$

Thus, required equation is

$$x^2 + (p + q - pq)x - pq\,(p+q) = 0.$$

Check Your Progress – 2

1. Let the roots be $\alpha - \beta, \alpha, \alpha + \beta$. Then

$(\alpha - \beta) + \alpha + (\alpha + \beta) = 48/32 \Rightarrow 3\alpha \Rightarrow = 3/2 \Rightarrow \alpha = ½$

$(\alpha - \beta)\alpha + \alpha(\alpha + \beta) + (\alpha - \beta)(\alpha + \beta) = 22/32$

$$\Rightarrow 3\,\alpha^2 - \beta^2 = \frac{21}{32}$$

$$\Rightarrow ¾ - \beta^2 = 22/32$$

$$\Rightarrow \beta^2 = \frac{3}{4} - \frac{22}{32} = \frac{2}{32} = \frac{1}{16}$$

$$\Rightarrow \beta = \pm\frac{1}{4}$$

Thus, required roots are

$$\frac{1}{2}-\frac{1}{4},\frac{1}{2},\frac{1}{2}+\frac{1}{4} \text{ or } \frac{1}{4},\frac{1}{2},\frac{3}{4}$$

These roots satisfy the relation product of roots = 3/32.

2. Let the roots be $\alpha/\beta,\ \alpha,\ \alpha\beta$

We have product of the roots $= \dfrac{8}{27}$

$$\Rightarrow (\alpha/\beta)\ (\alpha)\ (\alpha\beta) = \frac{8}{27}$$

$$\Rightarrow \alpha^3 = \frac{8}{27} \Rightarrow \alpha = \frac{2}{3}$$

Also,

$$\frac{\alpha}{\beta}+\alpha+\alpha\beta = \frac{-42}{27}$$

$$\frac{2}{3}\left(\beta+\frac{1}{\beta}\right)+\frac{2}{3} = -\frac{42}{27}$$

$$\Rightarrow \frac{2}{3}\left(\beta+\frac{1}{\beta}\right) = \frac{-14}{9}-\frac{2}{3} = \frac{-20}{9}$$

$$\Rightarrow \beta+\frac{1}{\beta} = \frac{-10}{3} \Rightarrow \beta = -3, -\frac{1}{3}$$

These values of α, β satisfy the relation

$$\frac{\alpha}{\beta}.\alpha + \alpha.\alpha\beta + \frac{\alpha}{\beta}.\alpha\beta = -\frac{-28}{27} \text{ (verify)}$$

3. Let the roots be $\dfrac{1}{\alpha-\beta},\dfrac{1}{\alpha},\dfrac{1}{\alpha+\beta}$

We have

$$\frac{1}{\alpha-\beta}.\frac{1}{\alpha}+\frac{1}{\alpha}.\frac{1}{\alpha+\beta}+\frac{1}{\alpha-\beta}.\frac{1}{\alpha+\beta} = \frac{12}{3}$$

and $\dfrac{1}{(\alpha-\beta)\alpha\ (\alpha+\beta)} = \dfrac{-4}{3}$

$$\Rightarrow \frac{(\alpha+\beta)+(\alpha-\beta)+\alpha}{\alpha(\alpha+\beta)(\alpha-\beta)} = 4$$

and $(\alpha-\beta)\alpha(\alpha+\beta) = \frac{-3}{4}$

Thus, 3 α = –3 ⇒ α = –1

Also, (–1–β) (–1) (–1+β) = –3/4

$$\Rightarrow 1-\beta^2 = \frac{3}{4} \quad \Rightarrow \beta^2 = \frac{1}{4} \Rightarrow \beta \pm \frac{1}{2}$$

$$\text{Thus, roots are } \frac{1}{-1-\frac{1}{2}}, \frac{1}{-1}, \frac{1}{-1+\frac{1}{2}}$$

or $-\frac{2}{3}, -1, 2$

It is easy to check these are roots of the equation.

4. Let the roots be α, β and γ. Suppose $\alpha + \beta = 1$

We have $\alpha + \beta + \gamma = 48/32 = 3/2 \Rightarrow$ r = ½

Also, $\alpha\beta + \beta\gamma + \gamma\alpha = 22/32$

$$\Rightarrow \alpha\beta + (\alpha+\beta)\gamma = \frac{22}{32}$$

$$\Rightarrow \alpha\beta + (1)\left(\frac{1}{2}\right) = \frac{22}{32} \Rightarrow \alpha\beta = 3/16$$

∴ equation whose roots are α and β is

$x^2 - (\alpha+\beta)x + \alpha\beta = 0$

$\Rightarrow x^2 - x + 3/16 = 0 \Rightarrow 16x^2 - 16x + 3 = 0$

$\Rightarrow 16x^2 - 4x - 12x + 3 = 0$

$\Rightarrow 4x(4x-1) - 3(4x-1) = 0$

$\Rightarrow (4x-3)(4x-1) = 0$

$\Rightarrow x = \frac{1}{4}, \frac{3}{4}$

It is easy to verify that $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ are roots of the equation.

5. Let the roots of the equation be 3α, 5α, and β.

   We have

   $3\alpha + 5\alpha + \beta = 9$ (i)

   $(3\alpha)(5\alpha) + (3\alpha)\beta + (5\alpha)\beta = 23$ (ii)

   and $(3\alpha)(5\alpha)\beta = 15$ (iii)

   From (i), $\beta = 9–8\alpha$

   Putting this in (ii) we get

   $15\alpha^2 + 8\alpha(9 - 8\alpha) = 23$

   $\Rightarrow \quad 49\alpha^2 - 72\alpha + 23 = 0$
   $\Rightarrow \quad 49\alpha^2 - 49\alpha - 23\alpha + 23 = 0$
   $\Rightarrow \quad 49\alpha(\alpha - 1) - 23\alpha(\alpha - 1) = 0$
   $\Rightarrow \quad (49\alpha - 1)(\alpha - 1) = 0 \Rightarrow = 1/49, 1$

   When $\alpha = 1, \beta = 1$

   Note that these values satisfy (iii)

   When $\alpha = 1/49$, $\beta = 9–8/49 = 433/49$

   These values do not satisfy the relation (iii)

   Thus, roots are 3,5, and 1

6. Let the roots be α, $\alpha+2$, and β

   We have

   $\alpha + (\alpha + 2) + \beta = 9$ (i)

   $\alpha(\alpha + 2) + \alpha\beta + (\alpha + 2)\beta = 23$ (ii)

   $\alpha(\alpha + 2)\beta = 15$ (iii)

   From (i) we get $\beta = 7 - 2\alpha$

   Putting this in (ii), we obtain

   $\alpha(\alpha + 2) + (\alpha + \alpha + 2)(7 - 2\alpha) = 23$

   $\Rightarrow \quad \alpha^2 + 2\alpha + 14 + 10\alpha - 4\alpha^2 = 23$
   $\Rightarrow \quad 3\alpha^2 - 12\alpha + 9 = 0 \Rightarrow \quad \alpha^2 - 4\alpha + 3 = 0$
   $\Rightarrow \quad (\alpha - 1)(\alpha - 3) = 0 \quad \Rightarrow \alpha = 1, 3$

   When $\alpha = 1, \beta = 5$

These values satisfy (iii)

Thus roots are 1,3,5

When $\alpha = 3$, $\beta = 1$

These values satisfy (iii)

In this case roots are 3,5, 1

7. Let the roots be $\alpha, \beta\, \gamma, \delta$

As these are in G.P.

$$\frac{\alpha}{\beta} = \frac{\gamma}{\beta} = \frac{\delta}{\gamma} \Rightarrow \gamma\delta = \beta\gamma$$

We have $\alpha + \beta + \gamma + \delta = \dfrac{195}{27}$ (i)

$$(\alpha + \delta)(\beta + \gamma) + \gamma\,\delta + \beta\,\gamma = \frac{494}{27} \quad \text{(ii)}$$

$$\gamma\,\delta\ \ (\beta + \gamma) + \beta\gamma\,(\alpha + \delta) = \frac{520}{27} \quad \text{(iii)}$$

$$(\alpha\,\delta)(\beta\gamma) = \frac{192}{27} \quad \text{(iv)}$$

As $\gamma\,\delta = \beta\,\gamma$, from (iv), we get

$(\gamma\delta)^2 = 64/9 \Rightarrow \gamma\,\delta = 8/3$ [Use (i) and (iii)]

Putting these values in (iii), we get

$(\alpha + \delta)\ (\beta + \gamma) = 350/27$ (v)

From (i) and (v), we get equation whose roots are $\alpha + \delta$ and $\beta + \gamma$, is

$$x^2 - \frac{195}{27}x + \frac{350}{27} = 0$$

or $27x^2 - 195x + 350 = 0$

$\Rightarrow 27x^2 - 90x - 105x + 350 = 0$

$\Rightarrow 9x(3x - 10) - 35x\,(3x - 10) = 0$

$\Rightarrow (9x - 35)\,(3x - 10) = 0$

$\Rightarrow x = \dfrac{35}{9}, \dfrac{10}{3}$

Let $\alpha + \delta = 35/9$, $\beta + \gamma = 10/3$

Equation whose roots are α, δ is

$$x^2 - \frac{35}{9}x + \frac{8}{3} = 0$$

$\Rightarrow \quad 9x^2 - 35x + 24 = 0$

$\Rightarrow \quad 9x^2 - 27x - 8x + 24 = 0$

$\Rightarrow \quad 9x(x-3) - 8(x-3) = 0$

$\Rightarrow \quad (9x-3)(x-3) = 0$

$$x = \frac{8}{9}, 3$$

Next equation whose roots β and γ is

$x^2 - (\beta + \gamma)x + \beta\gamma = 0$

$\Rightarrow \quad x^2 - \frac{10}{3}x + \frac{8}{3} = 0$

$\Rightarrow \quad 3x^2 - 10x + 8 = 0$

$\Rightarrow \quad (3x-4)(x-2) = 0$

$\Rightarrow \quad x = 4/3, 2$

Thus, roots are

8/9, 4/3, 2, 3 (verify)

8. Let the roots be

$\alpha, -\alpha, \beta$ and γ

We have

$\alpha + (-\alpha) + \beta + \gamma = 2/8 = 1/4$ (i)

$[\alpha + (-\alpha)](\beta + \gamma) + \alpha(-\alpha) + \beta\gamma = -27/8$ (ii)

$[\alpha + (-\alpha)]\beta\gamma + \alpha(-\alpha)(\beta + \gamma) = -6/8 = -3/4$ (iii)

$\alpha(-\alpha)\beta\gamma = 9/8$ (iv)

From (i) $\beta + \gamma = 1/4$

From (iii) $-\alpha^2(1/4) = -3/4 \Rightarrow \alpha^2 = 3 \Rightarrow \alpha = \pm\sqrt{3}$

From (ii) $-3 + \beta\gamma = -27/8 \Rightarrow \beta\gamma = -\frac{3}{8}$

Equation whose roots are β and γ is

$x^2 - (\beta + \gamma)x + \beta\gamma = 0$

or $\quad x^2 - (1/4)x - 3/8 = 0$

$\Rightarrow\ 8x^2 - 2x - 3 = 0$

$\Rightarrow\ x = 3/4, -1/2$

Thus, roots are $\sqrt{3}, -\sqrt{3}, \frac{3}{4}, \frac{-1}{2}$ (verify)

9. Let the roots be $\alpha, \beta\ \gamma$ and δ, and suppose that

$\alpha + \beta = \gamma + \delta$ (i)

We have

$\alpha + \beta + \gamma + \delta = -2$ (ii)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = -21$ (iii)

$(\alpha + \beta)\gamma\,\delta + \alpha\beta\,(\gamma + \delta) = 22$ (iv)

$\alpha\ \beta\ \gamma\ \delta = 40$ (v)

From (i) and (ii),

$\alpha + \beta = \gamma + \delta = -1$

From (iii), we get $(-1)(-1) + \alpha\beta + \gamma\,\delta = -21$

$\Rightarrow \alpha\beta + \gamma\delta = -22$

Equation whose roots are $\alpha\beta$ and $\gamma\delta\ is$

$x^2 - (\alpha\beta + r\,\delta)x + (\alpha\beta)(r\,\delta) = 0$

$\Rightarrow\ x^2 + 22x + 40 = 0$

$\Rightarrow\ (x + 2)(x + 20) = 0 \Rightarrow\ x = -2, 20$

Let $\alpha\beta = -2\ \gamma\ \delta = -20$

Equation whose roots are α, β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

or $x^2 + x - 2 = 0 \Rightarrow (x + 2)(x - 1) = 0 \Rightarrow x = -2, 1$

Equation whose roots are γ, δ

$x^2 - (\gamma + \delta)x + (\gamma\delta) = 0$

or $x^2 + x - 20 = 0 \Rightarrow (x + 5)(x - 4) = -4, 5$

$\Rightarrow\ x = -5, 4$

Thus, roots are $-2, 1, -5, 4$

10. Let roots be $\alpha, \beta\, \gamma$ and δ, and assume that

$\alpha\beta = \gamma\delta$ (i)

We have

$\alpha + \beta + \gamma + \delta = \frac{15}{2}$ (ii)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\,\delta = \frac{35}{2}$ (iii)

$(\alpha + \beta)\,\gamma\delta + \alpha\beta\,(\gamma + \delta) = \frac{30}{2} = 15$ (iv)

$\alpha\beta\gamma\delta = \frac{8}{2} = 4$ (v)

From (i) and (iv)

$\alpha\beta = \gamma\,\delta = \pm 2$

From (ii) and (iv), it follows that

$\alpha\beta = \gamma\,\delta = 2$

From (iii), $(\alpha + \beta)(\gamma + \delta) = \frac{35}{2} - 4 = 27/2$

Thus, equation whose roots are
$\alpha + \beta$ and $\gamma + \delta$ *is*

$$x^2 - (\alpha + \beta + \gamma + \delta)x + (\alpha + \beta)(\gamma + \delta) = 0$$

or $x^2 - (15/2)x + 27/2 = 0$ or $2\,x^2 - 15x + 27 = 0$

$\Rightarrow\ 2x^2 - 9x - 6x + 27 = 0 \quad \Rightarrow x(2x - 9) - 3(2x - 9) = 0$
$\Rightarrow\ (x - 3)(2x - 9) = 0 \Rightarrow x = 3, 9/2$

Let $\alpha + \beta = 3,\ \gamma + \delta = 9/2$

Equation whose roots are α and β is

$x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$x^2 - 3x + 2 = 0 \Rightarrow\ (x - 1)(x - 2) = 0$
$\Rightarrow x = 1, 2$

Equation whose roots are γ and δ is

$\Rightarrow\ x^2 - (\gamma + \delta)x + \gamma\,\delta = 0$

$\Rightarrow \quad x^2 - \left(\frac{9}{2}\right)x + 2 = 0$

$\Rightarrow \quad 2x^2 - 9x + 4 = 0$

$\Rightarrow \quad (2x - 1)(x - 4) - 0$

$\Rightarrow \quad x = ½, 4$

Thus, roots are

½, 1, 2, 4

11. Let roots be $\alpha, \beta\ \gamma$ and δ where $\alpha\beta = 13$ we have

$\alpha + \beta + \gamma + \delta = 10$ (i)

$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = 42$ (ii)

$(\alpha + \beta)\gamma\delta + \alpha\beta(\gamma + \delta) = 82$ (iii)

$\alpha\beta\gamma\ \delta = 65$ (iv)

As $\alpha\beta = 13$, from (iv) we get $\gamma\delta = 5$ from (iii)

$5(\alpha + \beta) + 13(\gamma + \delta) = 82$ (v)

From (i) and (v) $8(\gamma + \delta) = 32$

$\Rightarrow \quad \gamma + \delta = 4$

From (i) $\alpha + \beta = 6$

Equation whose roots are α and β is

$x^2 - 6x + 13 = 0 \ \Rightarrow \ x = 3 \pm 2i$

Equation whose roots are γ and δ is

$x^2 - 4x + 5 = 0 \ \Rightarrow \ x = 2 \pm i$

Thus, roots are $2 \pm i$, $3 \pm 2i$

12. Let the roots be $\frac{\alpha}{\beta}, \alpha, \alpha\beta$

We have

$\frac{\alpha}{\beta} + \alpha + \alpha\beta = -\frac{b}{a}$ (i)

$\frac{\alpha}{\beta}.\alpha + \alpha.\alpha\beta + \frac{\alpha}{\beta}.\alpha\beta = \frac{c}{a}$ (ii)

$$\frac{\alpha}{\beta}.\alpha\,.\,\alpha\beta \;=\; -\frac{d}{a} \qquad \text{(iii)}$$

(iii) ⇒ $\alpha^3 = -d/a$

From (i) $\alpha\,(\beta + \frac{1}{\beta} + 1) = -\frac{b}{a}$

and from (ii)

$$\alpha^2\,(\beta + \frac{1}{\beta} + 1) \;=\; \frac{c}{a}$$

$$\Rightarrow \alpha \;=\; -\frac{c}{b} \quad \Rightarrow \alpha^3 \;=\; \frac{-c^3}{b^3}$$

$$\therefore\; -\frac{d}{a} = -\,\frac{c^3}{b^3} \Rightarrow b^3 d \;=\; ac^3$$

13. Let roots be $\dfrac{1}{\alpha-\beta}, \dfrac{1}{\alpha}, \dfrac{1}{\alpha+\beta}$

We have

$$\frac{1}{\alpha-\beta} + \frac{1}{\alpha} + \frac{1}{\alpha+\beta} = p \qquad \text{(i)}$$

$$\frac{1}{(\alpha-\beta)\alpha} + \frac{1}{\alpha(\alpha+\beta)}\;\frac{1}{(\alpha-\beta)(\alpha+\beta)} = q \qquad \text{(ii)}$$

and $\dfrac{1}{(\alpha-\beta)(\alpha)(\alpha+\beta)} = r$ (iii)

From (i) and (iii), we get

and $\dfrac{(\alpha+\beta) + \alpha + (\alpha-\beta)}{(\alpha-\beta)(\alpha)(\alpha+\beta)} \;=\; q$

and $\dfrac{1}{\alpha-\beta(\alpha)(\alpha+\beta)} = r$

$\therefore\; 3\alpha r = q \quad \Rightarrow\; r = q/3r$

As $1/\alpha$ is a root of $x^3 - px^2 + qx - r = 0$,

We get

$$\frac{1}{\alpha^3} - \frac{p}{\alpha^2} + \frac{q}{\alpha} - r \;=\; 0$$

$\Rightarrow\ 1 - p\alpha + q\,\alpha^2 - r\,\alpha^3 = 0$

$$\Rightarrow\ 1 - \frac{pq}{3r} + q\,\frac{q^2}{qr^2} - r\frac{q^3}{27r^3} = 0$$

$\Rightarrow 27\,r^3 - 9pqr + 3q^3 - q^3 = 0$

$\Rightarrow\ 27\,r^3 - 9pqr + 2q^3 = 0$

14. We have

$\alpha + \beta + \gamma = p$

$\beta\gamma + \alpha\gamma + \alpha\beta = q$

$\alpha\beta\gamma = \gamma$

Now, $(\beta + \gamma)\,(\gamma + \alpha)\,(\alpha + \beta)$

$= (p - \alpha)\,(p - \beta)\,(p - \gamma)$

$= p^3 - pp^3 + qp - \gamma = pq - \gamma$

$$(ii)\quad \sum\frac{\alpha}{\beta} = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\alpha}{\gamma} + \frac{\gamma}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta}$$

$$= \frac{\beta + \alpha}{\alpha} + \frac{\gamma + \alpha}{\beta} + \frac{\alpha + \beta}{\gamma}$$

$$= \frac{p - \alpha}{\alpha} + \frac{p - \beta}{\beta} + \frac{p - \gamma}{\gamma}$$

$$= p\left(\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}\right) - 3$$

$$= \frac{pq}{r} - 3$$

$$\text{(iii)}\quad \sum\frac{1}{\alpha^2} = \frac{\beta^2\gamma^2 + \gamma^2\alpha^2 + \beta^2\alpha^2}{\alpha^2\beta^2\gamma^2}$$

$$= \frac{(\beta\gamma + \gamma\alpha + \alpha\beta)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)^2}{(\alpha\beta\gamma)^2}$$

$$= \frac{q^2 - 2rp}{r^2}$$

15. $\sum\alpha = -p, \sum\beta = q, \sum\alpha\beta\gamma = -r$ and

$\alpha\beta\gamma\delta = s$

Now,

(i) $\sum\alpha^2 = (\sum\alpha)^2 - 2\sum\alpha\beta$

$= p^2 - 2q$

$$(ii) \quad \sum\frac{1}{\alpha} = \frac{\sum\alpha\beta r}{\alpha\beta\gamma\delta} = \frac{-r}{s}$$

3.6 SUMMARY

This unit deals with solutions of equations of degrees 2, 3 and 4 in a single variable. In **sections 3.2 and 3.3**, first of all, the method of solving quadratic equations is given and then nature of these solutions is discussed. Next, method of forming quadratic equations for which the two roots are known, is discussed. In **section 3.4**, relations between roots and coefficients of cubic and biquadratic equations are mentioned. Using these relations, methods of solving special type of cubic and biquadratic equations are discussed. All the above-mentioned methods are illustrated with suitable examples.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 3.5**.