# UNIT 2 COMPLEX NUMBERS

## Structure



## 2.0 INTRODUCTION

All the numbers with which we have dealt so far were real numbers. However, some solutions in mathematics, such as solving quadratic equations require a new set of numbers. This new set of numbers is called the set of ***complex numbers***.

If we solve the equation $x^2 = 4$ for $x$, we find the equation has two solutions.
$x^2 = 4 \Rightarrow x = \sqrt{4} = 2$ or $x = -\sqrt{4} = -2$.

If we solve the equation $x^2 = -1$ in a similar way, we would expect it to have two solutions also.

$$x^2 = -1 \text{ should imply } x = \sqrt{-1} \text{ or } x = -\sqrt{-1}.$$

Each proposed solution of the equation $x^2 = -1$ involves the symbol $\sqrt{-1}$. For years it was believed that square roots of negative numbers denoted by $\sqrt{-5}, \sqrt{-2}$ and $\sqrt{-6}$ were nonsense. In the 17th century, these symbols were termed *imaginary numbers* by Rene Descartes (1596-1650). Now, the imaginary numbers are no longer thought to be impossible. In fact imaginary numbers have important uses in several branches mathematics and physics.

The number $\sqrt{-1}$ occurs so often in mathematics, that we give it a special symbol. We use better $i$ to denote $\sqrt{-1}$. Since $i$ stand for $\sqrt{-1}$, it immediately follows that $i^2 = -1$. The power of $i$ with natural exponent produces an interesting pattern, as follows :

$i^1 = i$, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, $i^6 = -1$, $i^7 = -i$, $i^8 = 1$

also $i^{-1} = -i$, $i^{-2} = -1$, $i^{-3} = i$, $i^{-4} = 1$

## 2.1 OBJECTIVES

After studying this unit, you will be able to :

- define complex number and perform algebraic operations such as addition, substraction, multiplication and division on the complex numbers;
- find modulus, argument and conjugate of a complex number;
- represent complex numbers in the argand plane;
- write polar form of a complex number;
- use Demoivre's theorem; and
- find cube roots of unity and verify some of the identities involving them.

## 2.2 COMPLEX NUMBERS

**Definition :** A *complex number* is any number that can be put in the form $a + bi$, where $a$ and $b$ are real number and $i = \sqrt{-1}$. The form $a + bi$ is called **standard form** for complex number. The number $a$ is called the real part of the complex number. The number $b$ is called imaginary part of the complex number.

We usually denote a complex number by $z$. We write $z = a + bi$. The real part of $z$ is denoted by Re ($z$) and the imaginary part of $z$ is denoted by Im($z$).

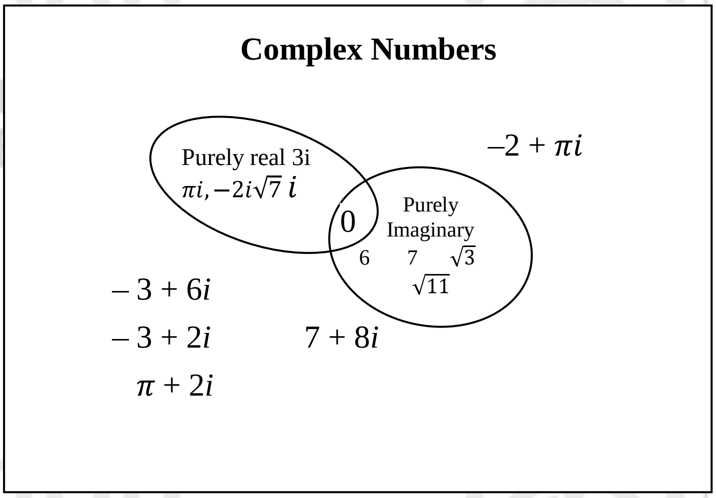


**Figure 1**

If $b = 0$, the complex number $a + bi$ is the real number $a$. Thus, any real number is a complex number with zero imaginary part. In other words, the set of real numbers is a subset of the set of complex numbers.

### Equality of two Complex Numbers

Two complex numbers are equal if and only if their real parts are equal and also their imaginary parts are equal.

Thus if, $z_1 = a + bi$ *and* $z_2 = c + di$ are two complex numbers, then $z_1 = z_2$, that is, $a + bi = c + di$ if and only if $a = c$ and $b = d$.

**Example 1** (*a*) Find $x$ and $y$ if $3x + 4\,i = 12 - 8y\,i$

(*b*) Find $a$ and $b$ if $(4a - 3) + 7\,i = 5 + (2b - 1)i$

**Solution :**

(a) Since the two complex numbers are equal, their real parts are equal and their imaginary parts are equal :

$$3x = 12 \text{ and } 4 = 8y \Rightarrow x = 4 \text{ and } y = -1/2$$

(b) The real parts are $4a - 3$ and 5. The imaginary parts are 5 and $2b - 1$.

$4a - 3 = 5$ and $7 = 2b - 1 \Rightarrow 4\,a = 8$ and $2b = 8 \Rightarrow a = 2$ and $b = 4$.

## 2.3 ALGEBRA OF COMPLEX NUMBERS

### Addition of two Complex Numbers

Two complex numbers such as $z_1 = a + bi$ and $z_2 = c + di$ are added as if they are algebraic binomials:

$$z_1 + z_2 = (a + bi) + (c + di) = (a + c) + (b + d)\,i$$

Observe that $a + bi = (a + 0i) + (0 + bi)$. In other words, $a + bi$ is the sum of the real number $a$ and the imaginary number $bi$.

Also observe that $z_1 + z_2$ is a complex number.

### Illustration

(i) $(3 + 4\,i) + (7 - 6\,i) = (3 + 7) + (4 - 6)\,i = 10 - 2\,i$

(ii) $(8 - 3\,i) + (6 - 2\,i) = (8 + 6) + (-3 - 2)\,i = 14 - 5\,i$

### Subtraction of Complex Numbers

If $z_1 = a + bi$ and $z_2 = c + di$, we define $z_1 - z_2$ as $z_1 + (-z_2)$.

That is, $z_1 - z_2 = (a + bi) + ((-c) + (-d)i) = (a - c) + (b - d)\,i$

**Example 2**

Fill in the blanks

(i) $(-4 + 10\,i) + (-1 + 2\,i) = \ldots.$ (ii) $(-6 + 17i) + (4 - 11\,i) = \ldots$
(ii) $(-4 + 2\,i) + (7 - 2\,i) = \ldots.$ (iv) $(3 - 5\,i) + (-3 + 5\,i) = \ldots$

**Solution**

(i) $-5 + 12i$ (ii) $-2 + 6i$
(iii) 3 (iv) 0

**Example 3**

Fill in the blanks

(i) $-(3 + 4i) = \ldots\ldots\ldots\ldots$ (ii) $(3 - 2i) - (4 - 3i) = \ldots\ldots\ldots\ldots..$
(iii) $(2 + 3i) - (i)$ ………….. (iv) $(5 + 2i) - 2 = \ldots\ldots\ldots\ldots$

**Solution**

(i) $-3 - 4i$ (ii) $-1 + i$
(iii) $2 + 2i$ (iv) $3 + 2i$

**Multiplication of Complex Numbers**

Two complex numbers such as $z_1 = a + bi$ and $z_2 = c + di$ are multiplied as if they were algebraic binomials, with $i^2 = -1$;

$$z_1\ z_2 = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

By definition, product of two complex numbers is again a complex number. Also observe that $yi = (y + 0i)\ (0 + 1i)$ and is, thus the product of the real number $y$ and the imaginary number $i$.

**Illustration 1**

$(3 + 2i)\ (4 + 5i) = 12 + 5i + 8i + 10i^2 = 12 + 15i + 8i - 10 = 2 + 23i$ $[\because i^2 = -1]$
and $(2 + 5i)(7 + 3i) = 14 + 6i + 35i + 15i^2 = 14 + 6i + 35i - 15 = -1 + 41i$
$[\because i^2 = -1]$

**Example 4** Perform the indicated operations and write the results in the form of $a + bi$

(i) $(2 + 3i)^2$ (ii) $(1 + i)^3$

(iii) $(\sqrt{5} + 7i)\,(\sqrt{5} - 7i)$

**Solution**

(i) $(2 + 3\,i)^2 = (2 + 3\,i)\,(2 + 3\,i) = (2)\,(2) + (2)\,(3i) + (2)\,(3i) + (3i)(3i)$

$$= 4 + 6\,i + 6\,i + 9\,i^2 = 4 + 12\,i - 9 = -5 + 12\,i$$

(ii) $(1+ i)^3 = (1+ i)\,(1+ i)\,(1+ i) = (1+ i + i + i^2)\,(1+ i) = (1+ i + i - 1)\,(1+ i)$

$$= 2\,i\,(1+ i) = 2\,i - 2\,i^2 = -2 + 2\,i$$

(iii) $(\sqrt{5} + 7\,i)\,(\sqrt{5} - 7\,i) = (\sqrt{5})(\sqrt{5}) - (\sqrt{5})(7i) + (\sqrt{5})(7i) - (7i)(7i)$

$$= 5 + 7\,(\sqrt{5}i) - 7(\sqrt{5}\,i) - 49\,i^2 = 5 + 49 = 54$$

Multiplicative Inverse of a Non-Zero Complex Number

If $a + i\,b \neq 0$ is any complex number, then there exists a complex number $x + iy$ such that

$$(a+ i\,b)\,(x + i\,y) = 1 + 0i = \text{the multiplicative identity in } C.$$

The number $x + i\,y$ is called the multiplicative inverse of $(a + ib)$ in $C$.

Now, $(a + ib)\,(x + iy) = 1 + 0i \Rightarrow (ax - by) + i\,(ay - bx) = 1 + 0i$

[multiplication of complex numbers]

$\Rightarrow$ $ax - by = 1$ and $ay + bx = 0$ [equality of two complex numbers]

$\Rightarrow$ $ax - by - 1 = 0$ and $ay + bx = 0$

Solving these equations for $x$ and $y$, we have

$$x = \frac{a}{a^2 + b^2} \qquad (1)$$

$$y = \frac{-b}{a^2 + b^2} \qquad (2)$$

both of which exist in **R,** because $(a + ib) \neq 0$ i.e., at least one of $a, b$ is different from zero.

Thus, the multiplicative inverse is of $a + ib$ is

$$x + iy = \frac{a}{a^2 + b^2} - i\,\frac{a}{a^2 + b^2} = \frac{a - ib}{a^2 + b^2}$$

Thus, every non-zero complex number has a multiplicative inverse in **C.**

**Division in Complex Numbers**

If $Z_1 = x + i\,y$ and $Z_2 = a + ib \neq 0$,

then

$$\frac{Z_1}{Z_2} = \frac{x + iy}{a + ib} = (x + iy)\,\frac{1}{(a + ib)}$$

$$= (x + i\,y)\,\frac{(a - ib)}{(a^2 + b^2)}$$

$$= \frac{ax + by}{a^2 + b^2} + i\,\frac{bx - ay}{a^2 + b^2}$$

**Example 5** If $\left(\frac{1 - i}{1 + i}\right)^{100} = a + ib,$ then show that $a = 1$ and $b = 0$

**Solution:** We have

$$\frac{1 - i}{1 + i} = \frac{(1 - i)}{(1 - i)}\frac{(1 - i)}{(1 + i)} = \frac{(1 - i)^2}{1^2 - i^2}$$

$$= \frac{1 - 2i + i^2}{2} = \frac{1 - 2i - 1}{2} = -i$$

Thus, $\left(\frac{1 - i}{1 + i}\right)^{100} = (-i)^{100} = 1$

$\therefore\ a + ib = 1 \Rightarrow a = 1$ and $b = 0$

**Example 6 :** If $x = -2 - \sqrt{3}\,i$, find the value of $2x^4 + 5x^3 + 7x^2 - x + 41$.

**Solution :**

$x = -2 - \sqrt{3}\,i, \Rightarrow x + 2 = -\sqrt{3}\,i \Rightarrow (x + 2)^2 = (-\sqrt{3}\,i)^2$

$\Rightarrow \quad x^2 + 4x + 4 = -3$ or $x^2 + 4x + 7 = 0$

We now divide $2x^4 + 5x^3 + 7x^2 - x + 41$ by $x^2 + 4x + 7$

$$
\begin{array}{r|l|l}
x^2 + 4x + 7 & 2x^4 + 5x^3 + 7x^2 - x + 41 & 2x^2 - 3x + 5 \\
 & 2x^4 + 8x^3 + 14x^2 & \\
 & \ominus \quad \ominus \quad \ominus & \\
\hline
 & -3x^3 - 7x^2 - x + 41 & \\
 & -3x^3 - 12x^2 - 21x & \\
 & \oplus \quad \oplus \quad \oplus & \\
\hline
 & 5x^2 + 20x + 41 & \\
 & 5x^2 + 20x + 35 & \\
 & \ominus \quad \ominus \quad \ominus & \\
\hline
 & 6 &
\end{array}
$$

Thus, $2x^4 + 5x^3 + 7x^2 - x + 41 = (x^2 + 4x + 7)(2x^2 - 3x + 5) + 6$

$= (0)(2x^2 - 3x + 5) + 6 = 6$

$\therefore$ value of $2x^4 + 5x^3 + 7x^2 - x + 41$ for $x = -2 - \sqrt{3}\, i$ is 6.

## Check Your Progress - 1

1. Is the following computation correct ?

$$\sqrt{-5}\,\sqrt{-7} = \sqrt{(-5)(-7)} = \sqrt{35}$$

2. Express each one of the following in the standard form $a + ib$.

(i) $\dfrac{1}{5-4i}$ (ii) $\dfrac{7+2i}{2-7i}$ (iii) $\dfrac{1}{\cos\theta + \mathrm{i}\,\mathrm{Sin}\,\theta}$ (iv) $\dfrac{2-\sqrt{-25}}{1-\sqrt{-16}}$

3. Find the multiplicative inverse of

(i) $\dfrac{1+i}{1-i}$ (ii) $(1 + \sqrt{3}\, i)^2$ (iii) $(1 + i)(1 + 2i)$

4. Find the value of $x^4 - 4x^3 + 4x^2 + 8x + 40$

when $x = 3 + 2i$.

5. If $(x + iy)^{1/3} = a + ib$, prove that

$$\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$$

6. Find the smallest positive integer for which

$$\left(\frac{1+i}{1-i}\right)^n = 1$$

## 2.4 CONJUGATE AND MODULUS OF A COMPLEX NUMBER

**Conjuate of a Complex Number**

**Definition :** If $z = x + i\,y$, $x, y \in$ **R** is a complex number, then the complex number $x - iy$ is called conjugate of $z$ and is denoted by $\bar{z}$.

For instance,

$$\overline{2+3i} = 2 - 3\,i\,,\ \overline{3-4i} = 3 + 4\,i,\ \bar{i} = -\,i \text{ and}$$

$$\bar{3} = \overline{3+0i} = 3 - 0i = 3$$

**Some properties of Complex Conjugates**

1. $\bar{\bar{z}} = z$
2. $\overline{z_1 z_2} = \bar{z}_1\ \bar{z}_2$
3. $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$
4. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}\ if\ z_2 \neq 0$

5 If $z = a + ib$, then

$z + \bar{z} = 2\,a = 2\,(z)$

and $z - \bar{z} = 2\,i\text{b} = 2i\ \text{Im}\,(z)$

6. $z = \bar{z}$ $\Leftrightarrow$ $z$ is real
7. $z = -\,\bar{z}$ $\Leftrightarrow$ $z$ is imaginary

**Modulus of a Complex Number**

**Definition :** If $z = x + iy$, $x, y \in$ **R** is a complex number, then the real number $\sqrt{x^2 + y^2}$ is called the modulus of the complex number $z$, and is denoted by $|z|$.

For instance, if $z = 2 + 3i$, then $|z| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}$

and if $z = 5 - 12i$, then $|z| = \sqrt{5^2 + (-12)^2} = \sqrt{25 + 144} = \sqrt{169} = 13$

Note that

$|z| = |-z| = |-\bar{z}| = |\bar{z}|$.

and if c is a real number, then $|cz| = |c|\,|z|$

**Some properties of Modulus of complex numbers**

1. $|z|^2 = z\,\bar{z}$
2. $|z| = 0 \Leftrightarrow z = 0$

3. $\frac{1}{z} = \frac{\bar{z}}{|z|^2}$ if $z \neq 0$

4. $|z_1 z_2| = |z_1|\,|z_2|$

5. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$ if $z_2 \neq 0$

6. $-|z| \leq z \leq |z|$

7. $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + \bar{z}_1 z_2$

$$= |z_1|^2 + |z_2|^2 + 2Re(z_1\bar{z}_2)$$

**Example 7:** If $a + ib \neq 0$, show that

$$\left|\frac{a-ib}{a+ib}\right| = 1$$

**Solution :** Let $z = a + ib$, then $\bar{z} = a - ib$

Since $|z| = |\bar{z}|$, we get

$1 = \frac{|\bar{z}|}{|z|} = \left|\frac{\bar{z}}{z}\right| = \left|\frac{a-ib}{a+ib}\right|$ $\left[\because \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}\right]$

**Example 8:** If $x + iy = \sqrt{\frac{a+ib}{c+id}}$, then $x^2 + y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$

**Solution :** $(x+iy)^2 = \frac{a+ib}{c+id}$

$\Rightarrow |(x+iy)^2| = \left|\frac{a+ib}{c+id}\right|$

$\Rightarrow |(x+iy)|^2 = \left|\frac{a+ib}{c+id}\right|$ $\left[\because \frac{|z_1|}{|z_2|} = \left|\frac{z_1}{z_2}\right|\right]$

$\Rightarrow \left(\sqrt{x^2+y^2}\right)^2 = \frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$

$\Rightarrow x^2 + y^2 = \sqrt{\frac{a^2+b^2}{c^2+d^2}}$

**Example 9:** If $(a - ib)(x + iy) = (a^2 + b^2)i$ and $a + ib \neq 0$, show that $x = b$ and $y = a$.

**Solution:** Let $z = a + ib$, then $z\bar{z} = a^2 + b^2$

Now, $(a+ib)(x - iy) = (a^2 + b^2)i$

$\Rightarrow z(x + iy) = z\bar{z}i$

$\Rightarrow x + iy = \bar{z}i = (a - ib)i = ai + b$

$\Rightarrow x = b,\ y = a$ [by definition of equality of Compex Numbers]

**Check Your Progress – 2**

1. Let $z = x + iy$ and $\omega = \frac{1 - iz}{z - i}$. If $|\omega| = 1$, show that $z$ is purely real.

2. If $|z| = 1$, $z \neq -1$ show that
$$\frac{z - 1}{z + 1} \text{ is purely imaginary}$$

3. If $|z - i| = |z + i|$, show that Im $(z) = 0$.

4. If $(a + bi)(3 + i) = (1 + i)(2 + i)$, find $a$ and $b$.

5. If $(\text{Cos}\,\theta + i\,\text{Sin}\,\theta)^2 = x + iy$, that show $x^2 + y^2 = 1$.

## 2.5 REPRESENTATION OF A COMPLEX NUMBERS AS POINTS IN A PLANE AND POLAR FORM OF A COMPLEX NUMBER

Let $OX$ and $OY$ be two rectangular axes in a plane with their point of intersection as the origin.

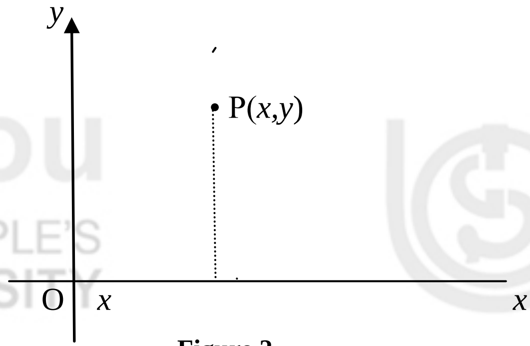


**Figure 2**

To each ordered pair $(x,y)$ there corresponds a point P in the plane such that the $x$-coordiante of P is $x$ and the $y$ – coordinate of $P$ is $y$. Thus, to a complex number $z = x + iy$ where corresponds a point $P\,(x,y)$ in the plane. Conversely, to every point $P'(x', y')$ there corresponds a complex number $x' + iy'$.

Thus, there is one-to-one correspondence between the set C of all complex numbers and the set of all the points in a plane.

For Example, the complex number $4 + 3i$ is represented by the point (4, 3) and the point $(-3, -4)$ represents the complex number $-3 - 4i$.

We note that the points corresponding to the complex numbers of the type $a$ lie on the $x$-axis and the complex numbers of the type $bi$ are represented by points on the $y$-axis.

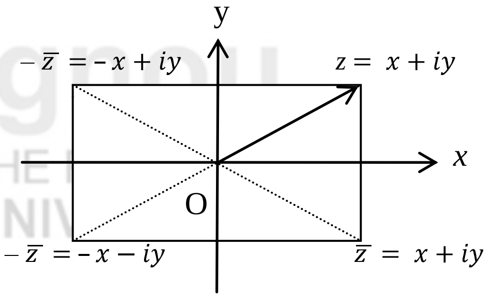


**Figure 3**

Note that the points $z$ and $-z$ are symmetric with respect to point $O$, while points z and $-\overline{z}$ are symmetric with respect to the real axis, since if $z = x + iy$, then $-z - (-x) + i(-y)$ and $\overline{z} = x + i(-y)$. See Figure 3.

**Remark :** Since the points on the $x$-axis represent complex number $z$ with $I(z) = 0$, the $x$-axis is also known as the real axis. Points on the y-axis represent complex numbers z with R($z$) = 0, the $y$-axis is also known as the imaginary axis. The plane is called as the *Argand plane, Argand diagram, complex plane or Gaussian plane.*

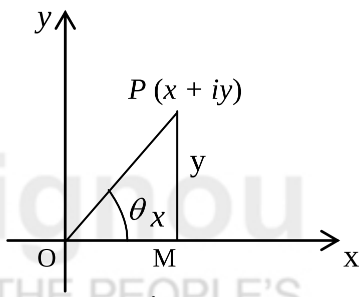


**Figure 4**

Note that $OP = \sqrt{x^2 + y^2} = |z|$

**Polar Representation of Complex Numbers**

Let $P(z)$ represents the complex number $z = x + iy$ as shown in the complex plane. Recall that the modulus or the absolute value of the complex number $z$ is defined as the length $OP$. It is denoted by $|z|$. Thus if $r = OP$; we have

$r = |z| = OP$

$= \sqrt{OM^2 + PM^2} = \sqrt{x^2 + y^2}$

$= \sqrt{[Re(z)]^2 + [Im(z)]^2}$

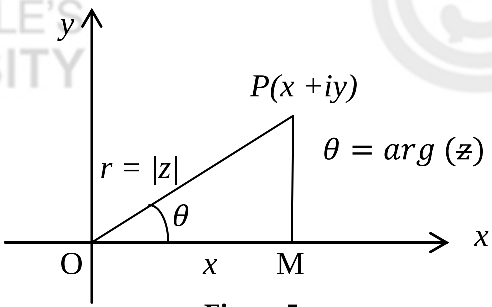


**Figure 5**

If $\theta$ be the angle which OP makes with OX in anticlockwise sense, then $\theta$ is called the *argument* or *amplitude* of the complex number $z = x + iy$.

Now in the right triangle $\Delta$OMP,

$x$ = OM = OP $\cos\theta = r\cos\theta$ (1)

$y$ = MP = OP $\sin\theta = r\sin\theta$ (2)

Thus, the complex number $z$ can be written as

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

This, is known as the *polar* form of the complex number.
Squaring and adding (1) and (2) we have

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2.1 = r^2$$

[Pythagorean identity]

Thus $r^2 = x^2 + y^2$ or $r = \sqrt{x^2 + y^2}$

which is the *modulus* of the complex number $z = x + iy$.

Dividing (2) and (1) , we have

$$\frac{y}{x} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta \Rightarrow \tan\theta = \frac{y}{x}.$$

$\theta$ is the argument of the complex number $z = x + iy$.

The value of $\theta$ $(-\pi < \theta \leq \pi)$ is called the *principal* value of the argument or amplitude of $z$. We denote it by Arg $z$ instead of arg $z$.

## 2.6 POWERS OF COMPLEX NUMBERS

**Product of *n* Complex Numbers**

We first take up product of complex numbers.

*If* $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$, $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, ………..
$z_n = r_n(\cos\theta_n + i\sin\theta_n)$, then

$$z_1 z_2 \ldots z_n = r_1 r_2 \ldots r_n[\cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)]$$

However, we shall not prove this statement.
When $r_1 = r_2 = \ldots\ldots\ldots\ldots r_n = 1$, we get

$$(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots\ldots\ldots\ldots(\cos\theta_n + i\sin\theta_n)$$
$$= \cos(\theta_1 + \theta_2 + \ldots\ldots\ldots\theta_n) + i\sin(\theta_1 + \theta_2 + \ldots\ldots\ldots\theta_n) \quad (1)$$

**Corollary** 1. $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$ and

2. $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$

**Proof** From (1), above we have

$\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)$

$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$= (\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)$

Equating real and imaginary parts, we get

$\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

and $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$

**De Moivre's Theorem (for Integral Index)**

Taking $\theta_1 = \theta_2 \ldots\ldots\theta_n = \theta$ in (1) we obtain

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

This proves the result for positive integral index.

However, it is valid for every integer *n*.

**Example 10 :** Use De Moivre's theorem to find $(\sqrt{3} + i)^3$.

**Solution** : We first put $\sqrt{3} + i$ in the polar form.

Let $\sqrt{3} + i = r(\cos\theta + i\sin\theta)$

$\Rightarrow \quad \sqrt{3} = r\cos\theta$ and $1 = r\sin\theta$

$\Rightarrow \quad (\sqrt{3})^2 + 1^2 = r^2(cos^2\theta + sin^2\theta)$

$\Rightarrow \quad r^2 = 4 \quad \Rightarrow \quad r = 2$

Thus, $\sqrt{3} + i = 2(cos\theta + i\sin\theta)$

$\Rightarrow \quad \sqrt{3} = 2\cos\theta$ and $1 = 2\sin\theta$

$\Rightarrow \quad 2\cos\theta = \frac{\sqrt{3}}{2}$ and $\sin\theta = \frac{1}{2}$

$\Rightarrow \quad \theta = 30°$.

Now, $(\sqrt{3} + i)^3 = [2\cos(30°) + i\sin(30°)]^3$

$= 2^3[\cos(30°) + I\sin(30°)]^3$

$= \quad 8\,[\cos(3 \times 30°) + i \sin(3 \times 30°))]$ [De Moivre's theorem]

$= \quad 8\,(\cos 90° + i \sin 90°) = 8(0 + i)$

$= \quad 8i$

**Cube Roots of Unity**

Let $x = (1)^{1/3}$

$\Rightarrow x^3 = 1 \Rightarrow\ x^3 - 1 = 0 \quad \Rightarrow\ (x-1)(x^3 +\ x + 1\,) = 0$

Therefore, either $x - 1 = 0\ \Rightarrow\ (x-1)(x^2 + x + 1) = 0$

$$\Rightarrow either\ x = 1\ or\ x = \frac{-1\ \pm\ \sqrt{(1-4)}}{2} = \frac{-1 \pm \sqrt{(-3)}}{2} = \frac{-1 \pm\ i\,\sqrt{(3)}}{2}$$

Thus, the three cube roots of unity are, $1, \dfrac{-1}{2} +\ i\ \dfrac{\sqrt{3}}{2}, \dfrac{-1}{2} - \dfrac{i\sqrt{3}}{2}$

Hence, there are three cube roots of unity.

Out of these one root (i.e., 1 ) is real and remaining two viz.,

$\dfrac{-1 + i\sqrt{(3)}}{2}\ and\ \dfrac{-1 - i\sqrt{(3)}}{2}$ are complex.

We ususally denote the cube root $\dfrac{-1}{2} + \dfrac{\sqrt{3}}{2}\ i$ by $\omega$ note that

$$\omega^2 = \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\ i\right)^2 = \frac{1}{4} - \frac{3}{4} - \frac{2\sqrt{3}}{4}\,\mathrm{i} = -\frac{1}{2} + \frac{\sqrt{3}}{2}\ i$$

Hence, the cube roots of unity are 1, $\omega$ , $\omega^2$.

Also, note that $\omega^3 = 1$.

**Some properties of Cube Roots of Unity**

1. $1 + \omega + \omega^2 = 0$
2. $\omega^3 = 1$
3. $\dfrac{1}{\omega} = \omega^2$ and $\dfrac{1}{\omega^2} = \omega$

**Example 11:** If 1, $\omega$, $\omega^2$ are cube roots of unity, show that

(i) $(1+\omega)^2 - (1+\omega)^3 + \omega^2 = 0$

(ii) $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11}) = 49$

**Solution** : (i) As $1+\omega+\omega^2 = 0$, we get

$1+\omega = -\omega^2$ and $1+\omega^2 = -\omega$

Thus,

$(1+\omega)^2 - (1+\omega^2)^3 + \omega^2$

$= (-\omega^2)^2 - (-\omega)^3 + \omega^2$

$= \omega^4 + \omega^3 + \omega^2 = \omega^3\omega + 1 + \omega^2$

$= \omega + 1 + \omega^2 = 0$

(ii) Since $\omega^{10} = (\omega^3)^3\,\omega = \omega$

and $\omega^{11} = (\omega^3)^3\,\omega^2 = \omega^2$,

Thus $(2-\omega)(2-\omega^2)(2-\omega^{10})(2-\omega^{11})$

$= (2-\omega)(2-\omega^2)(2-\omega)(2-\omega^2)$

$= [(2-\omega)(2-\omega^2)]^2$

$= [4 - 2\omega - 2\omega^2 + \omega^3]^2$

$= [4 - 2(\omega + \omega^2) + 1]^2$

$= [4 - 2(-1) + 1]^2$ $\quad [\because \omega + \omega^2 = -1]$

$= 7^2 = 49$

**Example 12:** If $x = a + b$, $y = a\omega + b\omega^2$

and $z = a\omega^2 + b\omega$, show that

$xyz = a^3 + b^3$

**Solution:** $xyz$

$= (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$= (a + b)(a^2\omega^3 + ab\omega^4 + ab\,\omega^2 + b^2\omega^3)$

$= (a + b)(a^2 + ab(\omega^3\omega + \omega^2) + b^2)$ $\quad [\because \omega^3 = 1]$

$= (a + b)[a^2 + ab(-1) + b^2)]$

$= (a + b)(a^2 - ab + b^2)$

$= a^3 + b^3$ $\quad [\because a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

**Check Your Progress – 3**

1. Calculate

   (i) $(\cos 30° + i \sin 30°)(\cos 60° + i \sin 60°)$
   (ii) $(\cos\theta + i \sin\theta)(\cos\theta - i \sin\theta)$
   (iii) $(\cos 45° + i \sin 45°)^2$

2. Use identities

   $\cos(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 sin\theta_2$

   $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 sin\theta_2$ to obtain values of

   (i) $\cos(75°)$ (ii) $\sin 75°$
   (iii) $\cos(90° + \theta)$ (iv) $\sin(90° + \theta)$
   (v) $\cos(105°)$ (vi) $\sin(105°)$

3. Using the identities in Question 2, show that

   $$tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - tan\theta_1\, tan\theta_2}$$

4. If 1, $\omega$, $\omega^2$ are three cube roots of unity, show that

   (i) $(1 + \omega)(1 + \omega^2)(1 + \omega^4)(1 + \omega^6)(1 + \omega^8) = 2$
   (ii) $(1 - \omega^2 + \omega^2)^5 + (1 - \omega^2 - \omega^2)^5 = 32$
   (iii) $(2 + 3\omega + 2\omega^2)^9 = (2 + 3\omega + 3\omega^2)^9 = 1$

5. If $x = a + b$, $y = a\omega + b\omega^2$ and

   $z = a\omega^2 + b\omega$, show that

   (i) $x + y + z = 0$ (ii) $x^2 + y^2 + z^2 = 6ab$

   (ii) $x^3 + y^3 + z^3 = 3(a^3 + b^3)$

## 2.7 ANSWERS TO CHECK YOUR PROGRESS

1. No.
   The formula $\sqrt{a}\sqrt{b} = \sqrt{ab}$
   holds when at least one of $a, b \geq 0$.

2. (i) $$\frac{1}{5 - 4i} = \frac{5 + 4i}{(5 - 4i)(5 + 4i)} = \frac{(5 + 4i)}{25 + 16}$$

   $$= \frac{5}{41} + \frac{4}{41}i$$

   (ii) $$\frac{7 + 2i}{2 - 7i} = \frac{7 + 2i}{-2i^2 - 7i} = \frac{7 + 2i}{(-i)(7 + 2i)} = \frac{-1}{i} = \frac{i^2}{i} = i$$

(iii) $\frac{1}{\cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{(\cos\theta + i\sin\theta)(\cos\theta - i\sin\theta)}$

$$= \frac{\cos\theta - i\sin\theta}{(\cos^2\theta - i^2\sin^2\theta)} = \frac{\cos\theta - i\sin\theta}{(\cos^2\theta + i^2\sin^2\theta)} = \cos\theta - i\sin\theta$$

(iv) $\frac{2-\sqrt{-25}}{1-\sqrt{-16}} = \frac{2-5i}{1-4i} = \frac{(2-5i)}{(1-4i)}\frac{(1+4i)}{(1+4i)}$

$$= \frac{2 - 5i + 8i - 20i^2}{1 - 16i^2}$$

$$= \frac{22+3i}{17} = \frac{22}{17} + \frac{3}{17}i$$

3. (i) Multiplicative inverse of $\frac{1+i}{1-i}$ is

$$\frac{1-i}{1+i} = \frac{1-i}{1+i}\frac{1-i}{1-i} = \frac{(1-i)^2}{1^2 - i^2} = \frac{1 + i^2 - 2i}{1+1} = \frac{1-1-2i}{2} = -i$$

(i) Multiplicative inverse of $(1+\sqrt{3}\,\mathrm{i})^2$is

$$\frac{1}{(1+\sqrt{3}\,i)^2} = \frac{(1-\sqrt{3}\,\mathrm{i})^2}{\left((1+\sqrt{3}\,i)\ (1-\sqrt{3}\,i)\ \right)^2} = \frac{1 - 2\sqrt{3}\,i + 3i^2}{(1+3)^2} = \frac{1 - 2\sqrt{3}\,i - 3}{16}$$

$$= \frac{-2-2\sqrt{3}\,i}{16} = -\frac{1}{8}(1+\sqrt{3}i)$$

(ii) We have

$(1+i)\,(1+2i) = 1 + 1\,i + 2\,i + 2\,i = 1 + 3\,i - 2 = -1 + 3i$

Its multiplicative inverse is

$$\frac{1}{-1+3i} = \frac{-1-3\,i}{(-1+3i)(-1-3i)}$$

$$= \frac{-1-3\,i}{1-9i^2} = \frac{-1-3\,i}{1+9} = -\frac{1}{10} - \frac{3}{10}\,i = -\frac{1}{10}(1+3i)$$

4. $x = 3 + 2\,i \Rightarrow x - 3 = 2\,i$

$\Rightarrow (x-3)^2 = (2\,i)^2 \Rightarrow x^2 - 6\,x + 9 = -4$

or $x^2 - 6\,x + 13 = 0$

Let's divide $x^4 - 4\,x^3 + 4\,x^2 + 8\,x + 39$ by $x^2 - 6x + 13$.

$$
\begin{array}{r|l}
x^2 - 6x + 13 & x^4 - 4x^3 + 4x^2 + 8x + 40 \quad \big( x^2 + 2x + 3 \\
 & x^4 - 6x^3 + 13x^2 \\
 & \ominus \quad \oplus \quad \ominus \\
\hline
 & 2x^3 - 9x^2 + 8x + 40 \\
 & 2x^3 - 12x^2 + 26x \\
 & \ominus \quad \oplus \quad \ominus \\
\hline
 & 3x^2 - 18x + 40 \\
 & 3x^2 - 18x + 39 \\
 & \ominus \quad \oplus \quad \ominus \\
\hline
 & 1
\end{array}
$$

Thus, $x^4 - 4x^3 + 4x^2 + 8x + 40$

$= (x^2 - 6x + 13)(x^2 + 2x + 3) + 1$

$= 0 + 1 = 1$

5. $x + iy = (a + ib)^3 = a^3 + i^3b^3 + 3a(ib)(a + ib)$

$= (a^3 - 3a^2b) + i(3a^2b - b^3)$

$\Rightarrow x = a^3 - 3a^2b \;\; and \;\; y = 3a^2b - b^3$

$\Rightarrow \frac{x}{a} = a^2 - 3b^2$ and $\frac{y}{b} = 3a^2 - b^2$

$\Rightarrow \frac{x}{a} + \frac{y}{b} = (a^2 - 3b^2) + (3a^2 - b^2) = 4(a^2 - b^2)$

6. We have $\frac{1+i}{1-i} = \frac{-i^2+i}{1-i} = \frac{i(1-i)}{1-i} = i$

$\left(\frac{1+i}{1-i}\right)^n = i^n$

$\therefore$ The smallest value of $n$ is 4.

**Check Your Progess – 2**

1. Let Ƶ = $x + iy$

Now, $|\omega| = 1 \Rightarrow |1 - i\,Ƶ| = |Ƶ - i|$

$\Rightarrow |1 - i(x + iy)| = |x + iy - i|$

$\Rightarrow |(1 + y) - ix| = |x + (y - 1)i|$

$\Rightarrow |(1 + y) - ix|^2 = |x + (y - 1)i|^2$

$\Rightarrow (1 + y)^2 + x^2 = x^2 + (y - 1)^2$

$\Rightarrow 1 + 2y + y^2 = y^2 - 2y + 1 \Rightarrow 4y = 0$ or $y = 0$

$\therefore$ Ƶ $= x \Rightarrow$ Ƶ is purely real.

2. Let Ƶ = $x + iy$

As |Ƶ| = 1, we get $x^2 + y^2 = 1$

Now, $\frac{z-1}{z+1} = \frac{(x-1)+iy}{(x+1)+iy}$

$= \frac{[(x-1)+iy][(x+1)-iy]}{(x+1)^2 + y^2}$

$= \frac{(x^2-1) + y^2 + iy\,(x+1-x+1)}{x^2 + 2x + 1 + y^2}$

$= \frac{(1-1) + 2i\,xy}{2(x+1)} = \frac{xy}{x+1}i$

$\Rightarrow \frac{z-1}{z+1}$ is purely imaginary.

3. Let $z = x + iy$

$|z - i| = |z + i|$

$\Rightarrow |x + iy - i| = |x + iy + i|$

$\Rightarrow |x + i\,(y - 1)|^2 = |x + i\,(y + 1)|^2$

$\Rightarrow \quad x^2 + (y-1)^2 = x^2 + (y+1)^2$

$\Rightarrow \quad (y-1)^2 - (y+1)^2 = 0$

$\Rightarrow -4y = 0 \Rightarrow y = 0$

Thus, Im $(z) = 0$

4. $a + bi = \frac{(1+i)(2+i)}{3+i} = \frac{2-1+3i}{3+i}$

$= \frac{1+3i}{3+i} = \frac{(1+3i)(3-i)}{(3+i)(3-i)}$

$= \frac{3+3+(9-1)i}{9+1} = \frac{6+8i}{10}$

$= \frac{3}{5} + \frac{4}{5}i \Rightarrow a = \frac{3}{5}, \quad b = \frac{4}{5}$

5. $|(\cos\theta + i\sin\theta)^2| = |x + iy|$

$|\cos\theta + i\sin\theta|^2 = |x + iy|$

$\Rightarrow |\cos\theta + i\sin\theta|^2 = \sqrt{x^2 + y^2}$

$\Rightarrow \left(\sqrt{\cos^2\theta + \sin^2\theta}\right)^2 = \sqrt{x^2 + y^2}$

$\Rightarrow x^2 + y^2 = 1$

**Check Your Progress – 3**

1. (i) cos (30° + 60°) + i sin (30° + 60°)

$= \cos 90° + \ i \sin 90° = i$

(ii) $(\cos\theta)^2 - i^2 \sin^2\theta = \sin^2\theta + \sin^2\theta = 1$

(iii) $\cos(2(45°) + i\sin(2(45°))$

$= \cos 90° + i \sin 90° = i$

2. (i) $\cos 75° = \cos(45° + 30°)$

$= \cos 45° \cos 30° - \sin 45° \sin 30°$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$$= \frac{(\sqrt{3}-1)\sqrt{2}}{4} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(ii) $\sin 75° = \sin(45° + 30°)$

$= \sin 45° \cos 30° - \sin 45° \sin 30°$

$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\frac{1}{2}$$

$$= \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{4}}{4}$$

(iii) $\cos(90° + \theta) = \cos 90° \cos\theta - \sin 90° \sin\theta$

$= (0)(\cos\theta) - (1)\sin\theta = -\sin\theta$

(iv) $\sin(90° + \theta) = \sin 90° \cos\theta + \cos 90° \sin\theta$

$= (1)(\cos\theta) + (0)\sin\theta = \cos\theta$

(v) $\cos(105°) = \cos(60° + 45°)$

$= \cos 60° \cos 45° - \sin 60° \sin 45°$

$$= \frac{1}{2}.\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} = -\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$$

$$= -\frac{\sqrt{6}-\sqrt{2}}{4}$$

(vi) $\sin(105°) = \sin 60° + 45°)$

$= \sin 60° \cos 45° (\cos \theta) + \cos 60° \sin 45°$

$$= \frac{\sqrt{3}}{2}.\frac{1}{\sqrt{2}} - \frac{1}{2}.\frac{1}{\sqrt{2}} = -\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

$$= -\frac{\sqrt{6}+\sqrt{2}}{4}$$

3. $\tan\left(\theta_1 + \theta_2\right) = \dfrac{sin(\theta_1 + \theta_2)}{cos(\theta_1 + \theta_2)}$

$$= \frac{\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2}{\cos\theta_1 \cos\theta_2 - \sin\theta_{1_1} \sin\theta_2}$$

Divide the numerator and denominator by $\cos\theta_1 \cos\theta_2$ to obtain

$$\tan(\theta_1 + \theta_2) = \frac{\tan\theta_1 + \tan\theta_2}{1 - \tan\theta_1 \tan\theta_2}$$

4. (i) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^6)(1+\omega^8)$

$= (1+\omega)(1+\omega^2)(1+\omega)(1+1)(1+\omega^2)$

$= 2((1+\omega)(1+\omega^2))^2 = 2((-\omega^2)(-\omega))^2 = 2\omega^6 = 2$

(ii) $(1-\omega+\omega^2)^5 + (1+\omega-\omega^2)^5$

$= (-\omega - \omega)^5 + (-\omega^2 - \omega^2)^5$

$= (-2)^5\omega^5 + (-2)^5(\omega^2)^5$

$= -32\omega^2 - 32\omega = -32(\omega^2+\omega)$

$= (-32)(-1) = 32$

(iii) $(2 + 3\omega + 2\omega^2)^9$

$= (2 + 2\omega + 2\omega^2 + \omega)^9 = (0+\omega)^9 = \omega^9 = 1$

and $(2 + 2\omega + 2\omega^2)^9 = (2 + 2\omega + 2\omega^2 + \omega^2)^9$

$= (0 + \omega^2)^9 = \omega^{18} = 1$

5. (i) $x + y + z = a(1+\omega^2+\omega)\, b(1+\omega^2+\omega)$

$= (0) + b(0) = 0$

(ii) $x^2 + y^2 + z^2$

$= (a^2 + b^2 + 2ab) + (a^2\omega^2 + b^2\omega^4 + 2ab\,\omega^3) + (a^2\omega^4 + b^2\omega^2 + 2ab\,\omega^3)$

$= a^2(1 + \omega^2 + \omega^2) + b^2(1 + \omega^4 + \omega^2) + 2ab(1 + \omega^3 + \omega^3)$

$= a^2(0) + b^2(0) + 2ab\,(1 + 1 + 1) = 6ab$

We know that

$x^3 + y^3 + z^3 - xyz$

$= (x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy)$

$= 0$

Thus, $x^3 + y^3 + z^3 = 3xyz.$

Also, $xyz = (a + b)(a\omega + b\omega^2)(a\omega^2 + b\omega)$

$= (a+b)\,[a^3\omega^3 + b^2\omega^3 + ab\,(\omega^2 + \omega^4)]$

$= (a+b)(a^2 + b^2 - ab) = a^3 + b^3$

Thus,

$$x^3 + y^3 + z^3 - 3xyz = 3\,(a^3 + b^3)$$

## 2.8 SUMMARY

In this unit, first of all, in **section 2.2**, the concept of complex number is defined. In **section 2.3**, various algebraic operations, viz., addition, subtraction, multiplication and division of two complex numbers are defined and illustrated with suitable examples. In **section 2.4**, concepts of conjugate of a complex number and modulus of a complex number are defined and explained with suitable examples. The properties of conjugate and modulus operations are stated without proof. In **section 2.5**, representation of a complex number as a point in a plane, in cartesian and polar forms, are explained. Finally, in **section 2.6**, DeMoivre’s Theorem for integral index, for finding nth power of a complex number, is illustrated with a number of examples. Also, some properties of cube roots of unity are discussed.

Answers/Solutions to questions/problems/exercises given in various sections of the unit are available in **section 2.7**.